Semi-empirical equations for the systematic decrease in permeability with depth in porous and fractured media

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Abstract Permeability loss with depth is a general trend in geological media and plays an essential role in subsurface fluid flow and solute transport. In the near surface zone where groundwater movement is active, the decrease in permeability with depth is dominated by the mechanical compaction of deformable media caused by the increase in lithostatic stress with depth. Instead of using empirical equations from statistical analysis, by considering the well-defined relationships among permeability, porosity, fracture aperture and effective stress under lithostatic conditions, new semi-empirical equations for the systematic depth-dependent permeability are derived, as well as the equations for the depth-dependent porosity in a porous medium and the depth-dependent fracture aperture in a fractured medium. The existing empirical equations can be included in the new equations as special cases under some simplification. These new semi-empirical equations perform better than previous equations to interpret the depth-dependent permeability of the Pierre Shale (with a maximum depth of approximately 4,500 m) and the granite at Stripa, Sweden (with a maximum depth of about 2,500 m).

Keywords Permeability · Porosity · Porous media · Fractured rocks · Sweden

Introduction

The decrease in permeability (as well as hydraulic conductivity) with depth has been documented for both porous media (e.g., Neuzil 1986; Whittemore et al. 1993; Hart and Hammon 2002; Wang et al. 2009) and fractured media (e.g., Snow 1968a; Louis 1974; Carlsson et al. 1983; Zhao 1998; Jiang et al. 2009a). For porous sedimentary rocks such as carbonate and sandstone reservoirs, the decrease in porosity with depth has been discussed in company with the depth-dependent permeability (Amthor et al. 1994; Budd 2001; Ehrenberg and Nadeau 2005). In fact, a unique relationship between permeability and depth is hardly observed in the field because permeability measurements at any depth show a remarkable variation. However, a systematic trend of decrease in porosity, hydraulic conductivity or permeability with depth can be commonly identified. Moreover, it is found that these trends are generally nonlinear.

The dependence of permeability on depth, which is a kind of large-scale permeability heterogeneity, plays an essential role in subsurface flow (Saar and Manga 2004). The pumping tests in chalk and limestone aquifers usually show an obvious decrease of yield with lowering of water level, which is dominated by variation of hydraulic conductivity with depth (Rushton and Chan 1976). In the investigation of groundwater discharge into rock tunnels, Zhang and Franklin (1993) found that a constant average hydraulic conductivity was unrealistic and the permeability gradient (the rate of decrease in permeability with depth) would significantly affect the discharge patterns. The effect of gradually decreasing hydraulic conductivity on water-table depths for steady-state subsurface drainage is also discussed by Gallichand (1994). In the study of topography driven groundwater flow, Marklund and Wörman (2007) found that the depth-dependent permeability controls the distribution of vertical flux driven by fluctuation of the landscape and would even affect the infiltration at ground surface. In sedimentary basins, the change in permeability with depth significantly affects the pore pressure distribution, which is significantly different from the results by using average permeability (Walder and Nur 1984; Bethke and Corbit 1988; Belitz and Bredehoeft 1988; Rice 1992; Wong et al. 1997).

A quantitative description of the permeability-depth relationship is necessary when the permeability loss is accounted for in numerical modeling of subsurface flow or in other involved research. Traditionally, empirical equations based on statistical analysis of permeability measurements are used to establish the permeability-depth correlation. In
particular, the (log-permeability)-depth relationship, which suggests that permeability decreases exponentially with depth, is the most frequently used model. The exponential correlation between permeability and depth was initially suggested by Louis (1974) and has been widely accepted (Zhang and Franklin 1993; Budd 2001; Marklund and Wörman 2007). This exponential equation for hydraulic conductivity is applied in one of the additional programs for Wörman 2007). This exponential equation for hydraulic conductivity is applied in one of the additional programs for MODFLOW-2000 (Anderman and Hill 2003). For porous media, similar exponential correlation is also applied between porosity and depth as Athy’s law (Athy 1930). The advantage of a simple exponential function is its convenience for analysis. However, in some cases, the simple exponential function fails to match field measurements and some non-exponential equations were employed. The (log-permeability)-(log-depth) relationship was proposed by Snow (1968b) to analyze permeability measurements of fractured crystalline rocks from several dam sites (with a maximum depth of 120 m). Manning and Ingebritsen (1999) also used the (log-permeability)-(log-depth) function to analyze the permeability loss with depth using permeability data of the continental crust inferred from geothermal data and progress of metamorphic reactions (with a maximum depth of 30 km). Another empirical function in the form of log(log-permeability)-depth, which implies that log-permeability decreases exponentially with depth, was suggested by Belitz and Bredhoef (1988) for sandstones in the Denver basin. Numerous factors such as tectonic activities, difference in lithology, thermal or chemical cementation and others would lead to permeability heterogeneity. It is difficult to handle the full effect of these geological processes in a single explicit equation. However, extracting a unique permeability-depth relationship dominated by some well-defined processes as a background of permeability heterogeneity is possible and expected. It is widely accepted that the gradual decrease in permeability (porosity) with depth can be described using the theory of hydro-mechanical coupling for both porous media (Neuzil 2003) and fractured media (Rutqvist and Stephansson 2003). As a result, hydro-mechanical coupling based models might be able to interpret the permeability-depth correlation in semi-empirical and semi-theoretical approaches. For porous media, theoretical solutions of porosity-depth relationship have been developed by using compaction models (Nagumo 1965; Ramm 1992; Connolly and Podladchikov 2000). However, to the authors’ knowledge, few similar studies have been done for permeability–depth relationship. Even if a theoretical solution of porosity-depth relationship can be applied to produce a permeability-depth relationship given a certain relationship between porosity and permeability for porous media, such an approach cannot satisfy fractured media, whose permeability is highly dependent on the special behavior of discontinuities. In this presentation, semi-empirical equations for the permeability-depth trend in porous media and fractured media are developed by considering the well-defined relationships among permeability, porosity, fracture aperture and effective stress under lithostatic conditions.

**The equations for permeability-depth in porous media**

For porous media (including the intact rock in fractured rock masses), the permeability is generally considered to be a result of the microstructure, characterized by the porosity, pore geometry, particle size (specific surface area) and other factors. These microstructure factors depend on the geological processes involved in forming the porous media and the depth-dependent stress condition. For the type of porous media formed under similar geological processes to those that result in similar distribution of solid particles (in this presentation, denoted as a porous medium), porosity can play the role of a dominant factor in controlling the permeability, which is relatively sensitive to buried depth. Compaction and aquathermal pressuring are considered to be two of the most important causes of porosity loss with depth (Luo and Vasseur 1992; Domenico and Schwartz 1998). Although aquathermal pressuring would be the essential cause of porosity loss for deep buried sediments, mechanical compaction is the primary factor controlling porosity for sediments near the ground surface. It is reported by Schneider et al. (1996) that the transition zone would be probably at a depth of a few hundred meters for carbonate sediments (Dunnington 1967; Beall and Fisher 1969) and around 1.5 km for sandstones (Fuchtbauer 1967; Schmidt and McDonald 1979; Bloch et al. 1986). Therefore, in the near-surface zone of the earth, where groundwater movement is most active, porosity-depth correlation can be analyzed based on mechanical compaction due to the change of effective stress.

In this section, a simplified model of depth-dependent permeability for porous media is developed based on the inter-relationship among permeability, porosity and mechanical properties under lithostatic stress. The objective is to find an equation that accounts for the most important factors causing the systematic permeability-depth trend in a porous medium. This equation should adopt the fundamental discoveries of relationships among permeability, porosity, effective stress and depth that can be described by functions as follows:

\[ k = F_{k,\phi}(\phi) \]  

(1)

\[ \phi = F_{\phi,\sigma}(\sigma'_m) \]  

(2)

\[ \sigma'_m = F_{\sigma,z}(z) \]  

(3)

where \( F \) indicates a function for relationship between two factors for a porous media, \( k \) is the permeability of porous media, \( \phi \) is the porosity, \( \sigma \) is stress, \( \sigma'_m \) is the mean effective
stress (positive for compaction), \( z \) is the depth. Then, the depth-permeability equation is derived as follows:

\[
\frac{dk}{dz} = \frac{dF_{k,0}}{d\phi} \frac{dF_{\phi,0}}{d\sigma_{\text{m}}} \frac{dF_{\sigma,\phi}}{dz} \tag{4}
\]

Widely accepted formulae for Eqs. (1)–(3) in the literature are applied in this presentation to find a semi-empirical solution for Eq. (4) which can interpret the general nonlinear systematic decrease of permeability with depth.

### The permeability-porosity relationship

Numerous approaches have been proposed to relate the permeability to porosity in porous media. Civan (2000) grouped these models into three categories: empirical equations, hydraulic tube models and network models. Hydraulic tube models are a reasonable compromise between the empirical and network models (Civan 2001). Among hydraulic tube models, the Kozeny-Carman model has been extensively used in the literature because of its simplicity (Bear 1972):

\[
k = F_{k,0}(\phi) = \frac{1}{S^2} \frac{\phi^3}{(1 - \phi)^2} \tag{5a}
\]

where \( S \) is the specific surface area of solid grains. Civan (2001) improved the Kozeny-Carman model by introducing non-integral exponents in Eq. (5a) to account for the interactive process between pore fluid and porous media. In some other classic models (e.g., Walsh and Brace 1984), permeability is proportional to integer powers of pore geometry parameters, i.e., porosity, hydraulic radius, tortuosity and/or specific surface area. According to these models, the permeability-porosity relationship can be simplified and described by the power-law (Bernabe et al. 2003):

\[
k = F_{k,0}(\phi) = k_0(\phi/\phi_0)^n \tag{5b}
\]

where \( k_0 \) is the reference permeability corresponding to the reference porosity, \( \phi_0 \), and \( n \) is a coefficient which is dependent on the grain size and the stack pattern of particles. Usually, the initial permeability, \( k_0 \), and initial porosity, \( \phi_0 \), are used for the reference permeability and reference porosity. It has been demonstrated that the exponent, \( n \), is also related to the ratio of effective to non-effective porosity (Bernabe et al. 2003; Ghabezloo et al. 2009). This non-effective porosity is dominated by dead-end pores (Bear 1972) filled by immobile fluid. It indicates a threshold or critical porosity for percolation in a porous medium.

Among the two permeability-porosity formulae, the power-law relationship seems more general with a variable factor, \( n \), for different types of materials. A compilation of laboratory data by David et al. (1994) shows that the value of \( n \) ranges from 4.6 to 25 in common geologic materials. In Lucia (1995), \( n \) is found to range from 4.275 to 8.537 for three ranges of permeability in sedimentary formations. In Ghabezloo et al. (2009), \( n \) is found to be 11 in a low permeability creeping material. However, some theoretical models suggest that \( n = 2 \) (Turcotte and Schubert 2001) or \( n = 3 \) (Zhu et al. 1995; Walsh and Brace 1984). For a given value of \( \phi_0 \), it is found that the Kozeny-Carman model is equivalent to a case of the power-law model with a special \( n \) value. As shown in Fig. 1a, when \( \phi_0 = 0.4 \), the curve of \( k/k_0 \) versus \( \phi/\phi_0 \) derived by Eq. (5a) is almost the same as that given by Eq. (5b) while \( n = 4 \). When \( \phi_0 = 0.2 \), this special \( n \) value is 3.4 as shown in Fig. 1b. It implies that the Kozeny-Carman model could be included by the power-law model.

In this study, it is assumed that the parameters, \( S \) in Eq. (5a) and \( n \) in Eq. (5b), for the permeability-porosity relationship are relatively stable in a porous medium. Thus permeability change in the porous medium can be described as a unique response of porosity change. From Eq. (5b) one has

\[
\frac{dF_{k,0}}{d\phi} = n(k_0/\phi_0^n)\phi^{n-1} = nk/\phi \tag{6}
\]

This is the first item prepared for Eq. (4). Note that the constant \( n \) value also implies that the ratio of effective to non-effective porosity (Bernabe et al. 2003; Ghabezloo et al. 2009) is assumed changeless in this study. Although Eq. (5b) is used in this paper, other models of permeability-porosity relationship can also be applied by similar steps.

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**Fig. 1** A comparison of the Kozeny-Carman model and the power law model of porosity-dependent permeability: a \( \phi_0 = 0.4 \), b \( \phi_0 = 0.2 \)
**Pore compressibility**

Compaction of porous materials, which leads to decrease of porosity, is controlled by the stress-strain behaviors. This compaction can be described by stress-dependent porosity. There are many different approaches to relate porosity change with stress. For example, Zimmerman et al. (1986) defined four compressibilities for porous rock, relating changes in the bulk and pore volumes to changes in the pore and confining pressures. A similar approach had been employed by Wang et al. (2009) to relate porosity with stress by defining bulk modulus of porous media and the matrix, respectively.

In this study, a different approach is used, i.e. the modified Athy’s law, which had been proposed by Davis and Davis (1999). The equation is written as follows:

\[
\phi = F_{\phi,\sigma}\left(\sigma'_m\right) = \phi_i + (\phi_0 - \phi_i)\exp\left(-\alpha\sigma'_m\right) \tag{7}
\]

where \(\phi_i\) is the reference porosity for zero effective stress (the initial porosity), \(\phi_r\) is the residual porosity at high stress, \(\alpha\) is a coefficient. A more simple equation can be written as:

\[
\phi = \phi_0\exp\left(-\alpha\sigma'_m\right) \tag{8}
\]

where the residual porosity is ignored, as adopted in Rubey and Hubbert (1959). Although Eq. (7) is empirical, it has been widely accepted (Rutqvist et al. 2002; Neuzil 2003; Liu et al. 2009). The significant advantage of this formula is inclusion of the residual porosity so that the “residual” permeability can be accounted for.

With the assumption of a constant \(\alpha\) value for a porous medium, Eq. (7) leads to

\[
\frac{dF_{\phi,\sigma}}{d\sigma'_m} = \frac{d\phi}{d\sigma'_m} = -\alpha(\phi - \phi_i) \tag{9}
\]

This is the second item prepared for Eq. (4).

**Effective stress under lithostatic conditions**

The action of effective stress on a porous medium is subject to the pressure of confining materials, porous fluid pressure and tectonic deformation. Measurements of the complete state of geostress have been conducted in many rocks at different depths across the world (Hoek and Brown 1980). It has been found that the lithostatic stress is generally proportional to depth.

In this simplified model, the relationship between the vertical effective stress, \(\sigma'_z\), the horizontal effective stress, \(\sigma'_H = \sigma'_x = \sigma'_y\), and the mean effective stress under a lithostatic condition are calculated with equations as follows:

\[
\sigma'_H = M\sigma'_z, \quad \sigma'_m = \frac{1}{3}\left(\sigma'_z + 2\sigma'_H\right) \tag{10}
\]

where \(M\) is the ratio of horizontal stress to vertical stress and is assumed to be constant. In addition, change of the vertical effective stress with depth, \(h\), is determined as

\[
d\sigma'_z = \gamma_e dz \tag{11}
\]

where \(\gamma_e\) is the effective unit weight which depends on density of the porous media and its saturation. In this presentation, the fully saturated situation and the equilibrium pore pressure are considered, thus \(\gamma_e\) can be calculated by

\[
\gamma_e = (\rho_s - \rho_w)g(1 - \phi) \tag{12}
\]

where \(\rho_s\) is the density of solid grains, \(\rho_w\) is the density of water, and \(g\) is the acceleration of gravity. Both \(\rho_s\) and \(\rho_w\) are assumed to be constant in this simplified model.

With Eqs. (10)—(12), change of the mean effective stress can be described as

\[
\frac{dF_{\sigma,\sigma}}{dz} = \frac{d\sigma'_m}{dz} = \gamma_e(1 - \phi) \tag{13}
\]

where \(\gamma_e = (1 + 2M)(\rho_s - \rho_w)g/3\) is a constant. This is the third item prepared for Eq. (4).

**The solution of depth-dependent permeability**

Substituting Eqs. (6), (9) and (13) into Eq. (4), the equation for depth-dependency of permeability can be rewritten as:

\[
dk = -n\gamma_e\alpha(\phi - \phi_i)(1 - \phi)k \tag{14}
\]

To solve this equation, the function of depth-dependent porosity, \(\phi(h)\), must be firstly derived. With Eq. (13), Eq. (9) can be rewritten as:

\[
\frac{d\phi}{dz} = -\gamma_e a(\phi - \phi_i)(1 - \phi) \tag{15}
\]

For constant values of \(\gamma_e\) and \(\alpha\), the solution of Eq. (15) is given by

\[
\phi(z) = \phi_i + \frac{(\phi_0 - \phi_i)(1 + \phi_i)}{(\phi_0 - \phi_i) + (1 + 2\phi_i - \phi_0)\exp[\gamma_e\alpha(1 + \phi_i)z]} \tag{16}
\]

where \(\phi_0\) is the initial porosity, which can be considered as the porosity on the ground surface.

Equation (16) is the simple semi-empirical solution of depth-dependent porosity for a porous medium. One can easily obtain the semi-empirical solution of the depth-

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dependent permeability by substituting Eq. (16) into Eq. (5a) or (5b). Using Eq. (5a), one has

\[
k(z) = k_0 \left(1 - \frac{1}{\phi_0} - \frac{1}{\phi_f} - \frac{1}{\phi_i} + \frac{1}{\phi_i} + \frac{1}{\phi_f} + \frac{1}{\phi_0} \right)^3 \left(1 - \frac{1}{\phi_0} - \frac{1}{\phi_f} - \frac{1}{\phi_i} + \frac{1}{\phi_i} + \frac{1}{\phi_f} + \frac{1}{\phi_0} \right)^2 \left(1 - \frac{1}{\phi_0} - \frac{1}{\phi_f} - \frac{1}{\phi_i} + \frac{1}{\phi_i} + \frac{1}{\phi_f} + \frac{1}{\phi_0} \right)
\]

(17a)

where \( k_0 \) is the initial permeability, which can be deemed as the permeability on the ground surface. Using Eq. (5b), one has

\[
k(z) = k_0 \left[ \frac{\phi_f}{\phi_0} + \frac{1}{\phi_i} + \frac{1}{\phi_f} + \frac{1}{\phi_i} + \frac{1}{\phi_f} + \frac{1}{\phi_0} \right]^n \left[ \frac{\phi_f}{\phi_0} + \frac{1}{\phi_i} + \frac{1}{\phi_f} + \frac{1}{\phi_i} + \frac{1}{\phi_f} + \frac{1}{\phi_0} \right]^2 \left[ \frac{\phi_f}{\phi_0} + \frac{1}{\phi_i} + \frac{1}{\phi_f} + \frac{1}{\phi_i} + \frac{1}{\phi_f} + \frac{1}{\phi_0} \right]
\]

(17b)

Equation 17b is the same solution of Eq. 15.

The equation for permeability-depth in a fractured medium

In fractured media, groundwater flow mainly takes place in discontinuities. Moreover, the occurrence of discontinuities can be the single most important factor that governs the mechanical properties as well as permeability of rock masses. Unlike porous media, the permeability of fractured rock is controlled by the aperture of individual fractures and their combination. Transmissivity of a fracture can be calculated by the famous cubic law with its aperture under a certain stress condition. Studies on the change of permeability in loading/unloading procedure for fractured rock masses have attracted numerous researchers. Equations of stress-dependent permeability are developed based on experimental studies and theoretical models for hydro-mechanical behavior of discontinuities (Rutqvist and Stephansson 2003). They encourage the attempt of expressing the permeability-depth trend with a simple equation for fractured rock masses (Snow 1968b; Oda 1986; Wei et al. 1995).

In this section, a new simplified model of depth-dependent transmissivity for a fracture is established and applied to give a systematic permeability trend of fractured media. This equation is developed on the fundamental discoveries of relationships among the transmissivity, the aperture, the effective normal stress and the depth of a fracture that can be described by functions as follows:

\[
T = G_{T,h}(b)
\]

(18)

\[
b = G_{b,\sigma}(\sigma'_n)
\]

(19)

\[
\sigma'_n = G_{\sigma,z}(z)
\]

(20)

where \( G \) indicates a function for relationship between two factors for a fracture, \( T \) is the transmissivity of a fracture, \( b \) is its aperture, \( \sigma'_n \) is the effective normal stress (positive for compaction). Then, the transmissivity-depth equation for a fracture is derived as follows:

\[
\frac{dT}{dz} = \frac{dG_{T,h}}{db} \frac{dG_{b,\sigma}}{d\sigma'_n} \frac{dG_{\sigma,z}}{dz}
\]

(21)

Widely accepted formulae for Eqs. (18)–(20) in the literature are adopted in this presentation to find a theoretical closed-form solution of Eq. (21). Subsequently, it is combined with considerations of the change in fracture frequency with depth to give an approximation of the average nonlinear depth-dependent permeability of a fractured rock.

The transmissivity of a fracture

Fluid flow in a deformable rock fracture can be described with the cubic law (Witherspoon et al. 1980) as follows:

\[
q = \frac{g(b_t + b_d)}{12\nu f} J = \frac{gb^3}{12\nu f} J
\]

(22)

where \( q \) is the volumetric flow rate through a joint per unit width, \( b_d \) is the apparent aperture, \( b_t \) is the residual aperture, \( J \) is the hydraulic gradient, \( \nu \) is the fluid viscosity and \( f \) is a friction factor that accounts for the roughness of the joint surface, \( b=b_t+b_d \) is the effective hydraulic aperture. Transmissivity, \( T \), of a fracture is generally defined as

\[
T = G_{T,h}(b) = \frac{q}{J} = \frac{gb^3}{12\nu f}
\]

(23)

Therefore

\[
\frac{dG_{T,h}}{db} = \frac{dT}{db} = \frac{gb^2}{4\nu f}
\]

(24)

This is the first item prepared for Eq. (21).

Discontinuity closure under stress

With increasing normal stresses on the faces, the aperture of a discontinuity would decrease. The apparent aperture can be interpreted as the residual aperture, \( b_n \) plus an apparent mechanical opening (Witherspoon et al. 1980)

\[
b_d = u_{\text{max}} - u_n
\]

(25)

Where \( u_n \) and \( u_{\text{max}} \) are the normal deformation of the fracture and the maximum closure, respectively.

Empirical models have been developed to give a correlation between the closure of a discontinuity and
the effective normal stress. The hyperbolic function proposed by Bandis et al. (1983) is most widely applied as

\[ u_n = \frac{\sigma'_n}{K_{n0} + \left(\frac{\sigma'_n}{u_{\text{max}}}\right)} \]  

(26)

where \( K_{n0} \) is the initial normal stiffness when \( u_n \) equals or is approximate to zero. \( K_{n0} \) belong to intrinsic mechanical properties of a discontinuity and can be considered as a constant. In this presentation, the initial condition is specified to the situation in which \( b_d = u_{\text{max}} \) and \( u_n = 0 \) with zero normal stress on the ground surface. If one defines \( b_0 = b_t + u_{\text{max}} \) as the maximum effective aperture of fracture at the initial condition, one has \( u_{\text{max}} = b_0 - b_t \) and \( u_n = b_0 - b \). So Eq. (26) can be rewritten into

\[ b = G_{b,\sigma} \left( \frac{\sigma'_n}{\sigma_n} \right) = b_0 - \frac{\sigma'_n}{K_{n0} + \left(\frac{\sigma'_n}{(b_0 - b_t)}\right)} \]  

(27)

which gives the stress-dependent effective aperture. For constant \( K_{n0} \), there is

\[ \frac{dG_{b,\sigma}}{d\sigma'_n} = \frac{db}{d\sigma'_n} = - \frac{(b - b_t)/(b_0 - b_t)}{K_{n0} + \left(\frac{\sigma'_n}{(b_0 - b_t)}\right)} \]  

(28)

This is the second item prepared for Eq. (21).

It is necessary to mention that laboratory investigations on single rock discontinuities had showed that both normal closure and shear dilation can change fracture transmissivity (Tsang and Witherspoon 1981; Raven and Gale 1985; Olsson and Barton 2001). However, when it comes to a rock section that contains a multitude of discontinuities, the change in transmissivity is mainly determined by the normal closure on fractures caused by normal stress (Barton et al. 1995; Min et al. 2004). Accordingly, as a reasonable approximation, the influence of shear stress on closure of a fracture is currently ignored in this model.

**Effective normal stress under lithostatic conditions**

Suppose that the fracture has a constant dip angle of \( \theta \), subjecting to the horizontal stress \( \sigma_1 \) and the vertical stress \( \sigma_3 \) (Fig. 2), the normal stress on the joint can be calculated by

\[ \sigma_n = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \cos 2\theta \]  

(29)

Under lithostatic conditions, applying Eqs. (10)–(12), the relationship between the effective normal stress and the depth can be described as

\[ \frac{d\sigma'_n}{dz} = \gamma_h = \left(\frac{1 + M}{2} - \frac{1 - M}{2} \cos 2\theta\right) \gamma_e \]  

(30)

\[ T(z) = T_0 \left[ 1 - \frac{\gamma_h z(b_0 - b_t)/(b_0 - b_t)}{(b_0 - b_t)K_{n0} + \gamma_h z} \right]^3 \]  

(35)

where \( T_0 \) is the transmissivity when \( b=b_0 \). Equation (35) is also the solution of Eq. (34).
To simplify the expression, defining a reference depth 
\[ z_c = (b_0 - b_i)K_{ni}/\gamma_b, \]
Eqs. (34) and (35) can be rewritten into
\[ b(z) = b_0 \left[ \frac{1}{1 + (z/z_c)} + \frac{(b_i/b_0)(z/z_c)}{1 + (z/z_c)} \right] \]  
(36)
\[ T(z) = T_0 \left[ \frac{1}{1 + (z/z_c)} + \frac{(b_i/b_0)(z/z_c)}{1 + (z/z_c)} \right]^3 \]  
(37)
Eqs. (36) and (37) are the simple analytical equations of 
the depth-dependent aperture and the depth-dependent 
transmissivity of a rock fracture, respectively.

The solution of depth-dependent permeability 
of a fractured rock mass

In a hydraulic test of a fractured rock, e.g., packer injection test, 
the test segment generally penetrates a group or several groups of joints. 
If the length of the segment is relatively short so an average depth can be 
applied and the flow in the hydraulic test is dominated by 
a group of parallel joints with the same aperture, 
the permeability of the test segment, \( k \), can be calculated by
\[ k = \frac{v}{g} \frac{NT}{L} = \frac{v}{g} \lambda T \]  
(38)
where \( N \) is the number of joints across the segment and \( L \) is 
the length of the segment, \( v/g \) is applied to transform the 
hydraulic conductivity to the permeability where \( v \) is the 
coefficient of kinematic viscosity, and \( \lambda = N/L \) is the depth-dependent 
density of the joints (fracture frequency). 
Assuming constant value of \( v/g \) and substituting Eq. (36) into Eq. (38), 
the depth-dependent permeability can be described by
\[ k(z) = k_0 \frac{\lambda}{\lambda_0} \left[ \frac{1}{1 + (z/z_c)} + \frac{(b_i/b_0)(z/z_c)}{1 + (z/z_c)} \right]^3 \]  
(39)
where \( k_0 \) and \( \lambda_0 \) is the permeability and the density of 
joints for the fractured rock on the ground surface, respectively.

According to Eq. 36, the fracture porosity of the rock mass due to opening of the joints can be approximate by
\[ \phi_i(z) = \lambda b \phi_{i0} \frac{\lambda}{\lambda_0} \left[ \frac{1}{1 + (z/z_c)} + \frac{(b_i/b_0)(z/z_c)}{1 + (z/z_c)} \right] \]  
(40)
where \( \phi_{i0} \) is the fracture porosity on the ground surface.

Equations. (39) and (40) give the general trends of permeability loss and fracture porosity loss of a fractured rock. They are quite different from the trends of permeability and porosity of a porous medium as described by 
Eqs. (16), 17a and 17b. However, a similar pattern can be seen, in that the permeability loss with depth is more significant than the porosity loss with depth. For the fractured rock, depth-dependent fracture density should be accounted for.

Discussion

Comparison of equations for porous media

In Connolly and Podladchikov (2000), an equation of depth-dependent porosity is derived as follows
\[ \phi(z) = \frac{\phi_0}{\phi_0 + (1 - \phi_0) \exp(\beta \Delta \rho g z)} \]  
(41)
where \( \Delta \rho g \) is defined similar to \( \gamma_c \) introduced in Eq. (13), 
\( \beta \) is the coefficient of pore compressibility. It is the 
pseudoelastic compaction profile as a typical result of a complex model in dealing with temperature-dependent viscoelastic compaction of sediments (Connolly and Podladchikov 2000). However, Eq. (41) is also closes to the special case of Eq. (16) if the residual porosity is ignored and \( \beta \) is equal to \( \alpha \). In this situation, the depth-dependent permeability of porous material can be simplified to
\[ k(z) = \frac{k_0}{[\phi_0 + (1 - \phi_0) \exp(\gamma_c \alpha z)]^{\pi}} \]  
(42)
Further simplification can be made by assuming a constant effective unit weight, \( \gamma_c \), in Eq. (12), i.e., \( \gamma_c = (\rho_s - \rho_w)g \). It leads to
\[ \frac{d\phi}{dz} = -\gamma_c \alpha (\phi - \phi_t) \]  
(43)
By solving Eq. (43), the depth-dependent porosity can be obtained as
\[ \phi(z) = \phi_t + (\phi_0 - \phi_t) \exp[-\gamma_c \alpha z] \]  
(44)
It is similar to the modified Athy’s law as shown in Eq. (7). If zero residual porosity is considered in Eq. (44), the widely used empirical exponential function of depth-dependent permeability can be derived as follows:
\[ k(z) = k_0 \exp[-n \gamma_c \alpha z] \]  
(45)
In Wang et al. (2009), the exponential function of depth-dependent porosity had also been derived by 
assuming a constant effective unit weight and a zero residual porosity. However, the solution of the depth-dependent permeability in Wang et al. (2009), which was obtained by the Kozeny-Carman model, is different from either Eq. (42) or Eq. (45). Instead, if zero residual porosity is assumed in Eq. (17a), which is also obtained.
by the Kozeny-Carman model, then Eq. (17a) can be simplified as

$$k(z) = k_0 \frac{\exp[-2\gamma_0\alpha z]}{\phi_0 + (1-\phi_0) \exp[\gamma_0\alpha z]}$$ (46)

As a typical example, the permeability-depth trend of the Pierre Shale is revisited. The data from Neuzil (1986) with a maximum depth of about 4,500 m is analyzed. The initial porosity of the shale approximates $\phi_0=0.4$ according to its deformation characteristics (Neuzil 1993). It is difficult to determine an accurate $n$ value for the shale if Eq. (17b) is applied. In this presentation, $n=4$ is chosen following Fig. 1a to make an agreement between the estimation and the observation data. As shown in Fig. 3, the systematic decrease of permeability of the Pierre Shale is well characterized by curves A and B with the initial permeability varying from $1.0 \times 10^{-4}$ millidarcies (mD) to $2.0 \times 10^{-7}$ mD. The confidence of permeability at a depth between curves A and B is 92.9%. An average trend is represented by curve C, which indicates that the residual permeability is $1.3 \times 10^{-6}$ mD on average. Note that curves A, B and C have different values of $\phi_r$. There are uncertainties if both the value of $\gamma_0\alpha(1+\phi_r)$ and the value of $n$ are unknown for the Pierre Shale. It is found that for any specific $n$ value between 1 and 5, one can search a corresponding value of $\gamma_0\alpha(1+\phi_r)$ to approximate the observation. The reason is that the permeability-depth trend given by Eq. (17b) is a coupling of the effects of $n$ and $\gamma_0\alpha(1+\phi_r)$. However, this uncertainty can not diminish the advantage of the semi-empirical equations in showing a similar pattern of systematic nonlinear decrease in permeability with depth as observed in the Pierre Shale.

Comparatively, the permeability-depth trend estimated by using Eqs. (42) and (46), i.e., with zero residual porosity, fails to approach a non-zero residual permeability as represented by curve D in Fig. 3. It shows a too rapid decrease below the reference depth (450 m) defined as the reciprocal of $\gamma_0\alpha(1+\phi_r)$. Curve E in Fig. 3 calculated using Eq. (45) has even more serious problem. It approximately coincides with curve C only in the extremely shallow part, where the change in unit weight is negligible.

**Comparison of equations for fractured media**

The empirically exponential function in describing decrease in permeability with depth for fractured rocks, which is widely used in the literature (Louis 1974; Zhang and Franklin 1993), can be interpreted as a special case of the new equation in this study. In Eq. (33), if the depth is significantly less than the reference depth and the residual aperture is ignored, there is

$$\frac{d \ln T}{dz} = -\frac{3}{z_c + z} \approx -\frac{3}{z_c}, z << z_c$$ (47)

It leads to

$$T(z) \approx T_0 \exp(-3z/z_c), z << z_c$$ (48)

Consequently, the exponential equation is an approximation for relatively shallow buried rock masses.

In Wei et al. (1995), hyperbolic equations have been proposed to describe the depth-dependent hydraulic aperture and permeability for a fractured rock mass as follows:

$$\frac{b}{b_0} = 1 - \frac{z}{A + Bz}$$ (49)

$$\frac{k}{k_0} = \left[1 - \frac{z}{A + Bz}\right]^3$$ (50)

where $A$ and $B$ are two constants. Oda (1986) derived similar equations with $B$ being equal to 1. One can see the relationship between Eqs. (48) and (36) if $A$ and $B$ are defined as

$$A = \frac{z_c}{1 - (b_z/b_0)}, B = \frac{1}{1 - (b_z/b_0)}$$ (51)

With these definitions, Eq. (36) can be rewritten as Eq. (48). Wei et al. (1995) estimated the data from Snow (1968a) and $A=58.0$ m and $B=1.02$ were obtained. This $B$
value indicates that \( b_r \) is about 2.0% of \( b_0 \) and \( z_c \) is 56.86 m in the study area of Snow (1968a). Wei et al. (1995) declared that the two constants can be fixed and applied to any rock masses.

In Eq. (39), which characterizes the depth-dependent permeability for fractured rock masses, fracture density is a pre-determined factor. It has been reported that discontinuity frequency would decrease with depth (Snow 1968b; Carlsson et al. 1983; Martin and Christiansson 2009; Jiang et al. 2009b). A high density of fractures near surface is generally caused by the effect of weathering and unloading. In this study, the data of granite at Stripa, Sweden, are selected for discussion. Decrease in fracture frequency with depth for the granite can be identified based on the data from four boreholes at Stripa (Carlsson et al. 1983) as shown in Fig. 4. However, a fractured zone exists 800–900 m below surface and results in an abnormally high fracture frequency. The decreasing trend of fracture frequency without the disturbed segment at 700–1,000 m depth can be approximated by

\[
\frac{\lambda}{\lambda_0} = 0.07 + 0.93 \exp(-0.0028z)
\]

where \( \lambda_0 \) is 5.6 m\(^{-1}\). This approximate trend is applied in this study to analyze the decrease in permeability with depth at Stripa. Wei et al. (1995) analyzed this depth-dependent permeability by using Eq. (50) without considering the fracture frequency. As previously mentioned, \( B = 1.02 \) indicates that \( b_r/b_0 = 0.02 \). However, by considering the decrease of fracture frequency as described with Eq. (51), different results of the permeability trend can be obtained by using the new equation, Eq. (39). It is found that \( b_r/b_0 = 0.08 \) satisfies the average permeability data from Carlsson et al. (1983) best while \( z_c \) is fixed at 56.86 m. As shown in Fig. 5, most of the data can be included between two curves of \( k_0 = 4 \) mD and \( k_0 = 25 \) mD.

For fractures in granite, the residual aperture obtained by Witherspoon et al. (1980) is 3.2–13.1 µm which is 3–12% of the initial aperture. Accordingly, \( b_r/b_0 = 0.02 \) seems too small and \( b_r/b_0 = 0.08 \) is more confident. However, more work is needed in study of the in situ characteristics of fractures at Stripa to check the efficiency of the alternative equations. Figure 5 also shows that in the shallow part (less than 500 m), the curve by Wei et al. (1995) and the curve generated using Eq. (39) coincides with each other.

**Remarks on limitations**

The new semi-empirical equations of the depth-dependent permeability, Eqs. (17a) and (17b) for porous media, and Eq. (39) for fractured media, are constrained by the simplifications in setup of the models and limited by the efficiency of the adopted formulas in describing the hydro-mechanical coupling processes. They are new approximations of the features that control the variation in permeability with depth.

Firstly, the equations give unique systematic depth-dependent permeability due to hydro-mechanical coupling for a single porous medium and fractured medium, without consideration of the change in lithology. They may well satisfy the measured permeability at a site where deep buried sediments or rocks are similar to that near surface. However, different parameters are required if the lithology of the medium significantly changes with depth. Faults or fractured zones developed by tectonic movements can also cause nonsystematic change of the permeability for the same fractured rock mass.

The lithostatic stress and hydrostatic fluid pressure conditions assumed in the models would probably be false if abnormal tectonic stress or fluid pressure is significant. Abnormal high or low fluid pressure can be induced by

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**Fig. 4** The average fracture frequency versus depth at Stripa, Sweden (data from Carlsson et al. 1983)

**Fig. 5** Change of permeability with depth at Stripa, Sweden (data from Carlsson et al. 1983). The hydraulic conductivity is transformed into the permeability with \( v/g = 10^{-7} \) m/s according to a normal condition
deposition or erosion in a sedimentary basin (Jiao and Zheng 1998). In this situation, additional trends of permeability change with depth over the systematic permeability-depth relationship are needed. Temporal variations in total stress and fluid pressure can also occur in a sedimentary basin that will lead to time-dependent change in permeability with depth. It is not handled in the new equations in the current paper. However, for a relatively short period during a geological process, this time-dependent effect can be generally ignored.

Equations (17a) and (17b) are based on the Kozeny-Carman model and power-law model, respectively, in describing the permeability-porosity relationship. In derivation of the equations, the specific surface area of the particles, $S$, and the exponent, $n$, are assumed to be constants. It is possible that change of these items with depth can also influence the permeability-depth trend. Generally, the compressibility of the solid grains in a porous medium is smaller than that of the framework. As a result, change of specific surface area would not be a serious problem. For unconsolidated fine grained sediments, it is possible that the microstructure of the pore such as its connectivity, space tortuosity and the amount of dead-end pores (non-effective porosity), will significantly depend on the stress condition and affect the depth-dependent permeability. However, few theoretical or empirical formulas have been proposed that account for this process. This is a topic that deserves further investigation.

In particular, Eqs. (39) and (40) satisfy an individual fracture set with similar dip angle and mechanical properties. They are problematic while the reference depth, $z_c$, and the relative residual aperture, $b_f/b_0$, differ for different groups of fractures. However, if it is necessary, the permeability tensor can be produced by a combination of the individual fracture sets obeying Eq. (39). Identification of different parameters for the fracture sets is difficult with the normal hydraulic test method in boreholes. In this situation, the estimated $z_c$ and $b_f/b_0$ within a single equation, represent average behavior of the rock fractures, which dominate the flow in a hydraulic test. In Eq. (30), $\eta_b$ is independent of the dip angle when $M=1$. This is generally approximated under lithostatic conditions and encourages the use of Eq. (39) in describing the average permeability-depth trend of fractures with variable direction but similar mechanical properties.

The semi-empirical equations are available only in the surficial environment where permeability loss is dominated by gravitational compaction and the effect of aquathermal processes such as chemical cementation, is insignificant. In this presentation, the new permeability-depth trend equations show their efficiency for a porous medium with depth less than 4,500 m (Fig. 3) and for a fractured medium with depth less than 2,500 m (Fig. 5). Manning and Ingebritsen (1999) reports the permeability of the continental crust with a maximum depth of 30 km in which the (log-permeability)-(log-depth) equation ($\log k = 14.32 + 0.83 \log z$) is applied. Due to complexity in rock types and mixing of porous and fractured media as well as a significant change of tectonic and geothermal conditions in this large scale of depth, it is hard to conclude that this log-log trend can well represent the permeability-depth trend for a porous medium or a fractured rock in near surface conditions. In fact, the Manning and Ingebritsen (1999) equation fails to match the observations shown in Figs. 3 and 5. However, one can see an approximate log-log trend in a certain depth range with the semi-empirical equations reported here, as characterized by the linear part of the curves in Fig. 3 (depth range: 200–1,000 m) and Fig. 5 (depth range: 100–1,000 m).

Conclusions

Nonlinear decrease in permeability with depth is a general trend in both porous and fractured media. In the near surface zone where groundwater movement is active, it is dominated by the hydro-mechanical coupling of deformable media with increasing stress under gravity loading. New semi-empirical equations that express the systematic depth-dependent permeability of a porous/fractured medium are presented in this study.

Instead of using empirical equations based on statistical analysis, by considering the well-defined relationships among permeability, porosity, and effective stress under the lithostatic condition, a semi-empirical model is proposed to describe the unique change in permeability with depth for a porous medium. A new equation of depth-dependent permeability is derived as well as a new equation of depth-dependent porosity. The empirical exponential function and the equation developed by Connolly and Podladchikov (2000) can be considered as specially simplified cases of the new equation. In interpreting the permeability trends of the Pierre Shale (Neuzil 1986), with maximum depth approximating 4,500 m, the new equation performs better than the previous equations.

For fractured rock, a model of depth-dependent transmissivity of a fracture is presented, which considers the cubic law, the closure of the fracture aperture under the lithostatic condition. A widely accepted formula for the relationship between aperture and normal effective stress is applied to handle the deformation behavior of the fracture. A closed-form semi-analytical solution of the transmissivity-depth relationship for a fracture is derived as well as the solution of depth-dependent aperture. The solution is combined with the consideration of fracture frequency to produce the equation of systematic depth-dependent permeability for a fractured medium. The empirical exponential function and the equation developed by Wei et al. (1995) and Oda (1986) can be considered as specially simplified cases of the new equation. Comparable application results of the equations are presented in interpreting the permeability trends of the granite at Stripa, Sweden, with a deepest depth of approximately 2,500 m.

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