Type Curves Analysis for Asymmetrically Fractured Wells

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In this paper, a new constant rate solution for asymmetrically fractured wells was proposed to analyze the effect of fracture asymmetry on type curves. Calculative results showed that for a small wellbore storage coefficient or for the low fracture conductivity, the effect of fracture asymmetry on early flow was very strong. The existence of the fracture asymmetry would cause bigger pressure depletion and make the starting time of linear flow occur earlier. Then, new type curves were established for different fracture asymmetry factor and different fracture conductivity. It was shown that a bigger fracture asymmetry factor and low fracture conductivity would prolong the time of wellbore storage effects. Therefore, to reduce wellbore storage effects, it was essential to keep higher fracture conductivity and fracture symmetry during the hydraulic fracturing design. Finally, a case example is performed to demonstrate the methodology of new type curves analysis and its validation for calculating important formation parameters.

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Keywords: asymmetrically fractured wells, fracture conductivity, type curves, fracture asymmetry factor, formation parameters

1 Introduction

During the last few decades, a number of well test analysis methods of fractured wells have been proposed [1–4] in the determination of formation properties. Gringarten [5] made an extraordinary contribution to the development of transient pressure analysis and type curves analysis of fractured wells. To consider fracture conductivity effects, a semi-analytical solution and an analytical solution [6,7] for a vertical fractured well were presented. Later several authors [8–21] applied the solutions to the analysis of productivity and well test data for fractured wells. The previous theoretical analysis of fractured wells is based on the same assumption postulating a symmetrically homogeneous fracture. However, actual fractures are asymmetric and there are only a few papers at present researching asymmetric fractures.

Crawford and Landrum [22] first studied the problem of fracture asymmetry, but did not draw meaningful conclusions. Until 1979 the effect of fracture asymmetry on the pressure behavior of fractured wells at constant rate was discussed by Narasimhan and Plen [23]. Later, Bennet [24] briefly analyzed the influence of the fracture asymmetry on production. However, all these authors studied asymmetric fracture by means of numerical simulation methods.

Resurreic and Fernando [25] obtained a semi-analytical solution in real space under the constant pressure condition. They found that in the process of the flow of fluid, the reciprocal of rate would be seriously affected by asymmetry factor when fracture asymmetry factor \(a_s \geq 0.8\) and the impact would be greater when \(a_s = 1\).

Rodriguez et al. [26] established a new semi-analytical model of finite conductivity asymmetric fracture using Green’s functions, and obtained a semi-analytical solution under the constant rate condition. According to the presented solution, he found the curves of \(C_{Dp}p_{wD}(1 + 3a_s^2)\) versus \(C_{Dp}^{1/2}(1 + 3a_s^2)\) would show a linear relationship in the Cartesian coordinate system, and the intercept \(b_D(a_s, C_{Dp})\) is a function of dimensionless conductivity and asymmetry factor.

Berumen et al. [27] proposed constant rate solutions for a fractured well with an asymmetric fracture by employing numerical simulation methods. And then, he made a series of type curves of the ratio of pressure and pressure derivative under different \(a_s: \log–\log\) curves of \(p_D/(2\pi t_D)\) versus \(t_D (1 + a_s) 2C_{Dp}^2\). Since they were numerically solved, the results had a certain error and the application of the curves was not given by the authors.

Recently, Tiab et al. [16] reported an interesting paper, in which Tiab’s direct synthesis (TDS) technique was applied to the bilinear flow model and linear flow model to calculate dimensionless conductivity and asymmetry factor. Meanwhile a linear pressure equation which was simple and practical considering both fracture conductivity and asymmetry factor was developed by means of regression analysis method.

However, their models proposed by the authors above can’t be applied to well test analysis without considering the wellbore storage and skin effects. The objective of this paper is to establish a new model for well test analysis of a finite conductivity asymmetric fracture including the wellbore storage and skin effects. New solutions for asymmetrically fractured wells under the constant rate condition are presented using Laplace transform and point source integral methods. Compared to the previous solutions [25–27], the new solution has several advantages. First of all, the new solution is obtained in Laplace space, so it not need to scatter time, further reducing the amount of computation and improving the computational efficiency; Second, a constant pressure solution can be calculated directly from a constant rate solution and thirdly it is convenient for us to add the wellbore storage effect into the constant rate solution.

2 Pseudolinear Flow Model

2.1 Define Dimensionless. Dimensionless pressure, dimensionless rate, dimensionless time, dimensionless conductivity is defined as follows [12,16,28]

\[
p_{wD} = \frac{kh}{141.2\mu B} \Delta p_w
\]

\[
t_D = \frac{0.002637\kappa \sqrt{t}}{\phi \mu c_s
\]

\[
C_{ID} = \frac{k_i W_i}{k x_i}\]

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2.2 Pseudolinear Flow Model Under Constant Rate Condition. Cinco and Meng [29] developed a pseudolinear flow solution for symmetrically fractured wells. According to his method, a pseudolinear solution for asymmetrically fractured wells in Laplace domain was given as [26]

\[ \bar{p}_{wD} = \sqrt{\frac{2\pi}{C_{ID}}}(s^{-5/4}) \left\{ \tanh \left[ 1 + a_s \frac{s^{1/4}}{\sqrt{\frac{2}{C_{ID}}}} \right] \right\}, \]

\[ + \tanh \left[ 1 - a_s \frac{s^{1/4}}{\sqrt{\frac{2}{C_{ID}}}} \right]^{-1} \]  \tag{1}

When \( t_0 \) is small (large \( s \)), Eq. (1) can be simplified as

\[ \bar{p}_{wD} = \sqrt{\frac{2\pi}{C_{ID}}}(s^{-5/4}) \]  \tag{2}

A pressure solution for an asymmetric fracture in the period of bilinear flow through inversion of Eq. (2) can be obtained as follows:

\[ p_{wD} = \frac{\pi}{\Gamma(5/4)\sqrt{2C_{ID}}} t_0^{1/4} \]  \tag{3}

When \( t_0 \) is large (small \( s \)), Eq. (1) can be simplified as

\[ p_{wD} = \frac{\pi}{2} s^{3/2} + \frac{\pi(1 + 3a_s^2)}{3C_{ID}} s^{-1} \]  \tag{4}

A constant pressure solution for an asymmetric fracture through inversion of Eq. (4) in the period of linear flow can be given as

\[ p_{wD} = \sqrt{\pi t_0} + \frac{\pi(1 + 3a_s^2)}{3C_{ID}} \]  \tag{5}

3 A New Semi-Analytical Model

Rodriguez [26] proposed a mathematical model for an asymmetric fracture in an infinite reservoir and a semi-analytical solution in Laplace domain was given as [26,27]. The definitions of Eqs. (6)–(8) are the same with the literatures [26,27]. \( \theta \) in Eq. (6) is the asymmetry factor, which is a measure of distance from the position of the well to the center of the fracture, and it is defined as follows

\[ \theta = \frac{x_w}{x_i} \]  \tag{9}

We could give the Eqs. (6)–(8) a solution of Laplace domain by using Green’s functions and Laplace transform methods, which is the following (see Appendix A)

\[ \bar{p}_{wD}(x_D,s) = \left[ \bar{p}_{wD}(s) \right]_{avg} + \frac{\pi}{C_{ID}} \int_{x_D}^{x} N(x',x_D) \bar{q}_{wD}(x',s) dx' \]

\[ - \frac{2\pi}{C_{ID}} S_k N(\theta, x_D) \]  \tag{10}

In Eq. (10), \( \left[ \bar{p}_{wD} \right]_{avg} \) is the average pressure for any time in the fracture, \( N(x',x_D) \) is the second Green’s function, which is a piecewise function and is defined as

\[ N(x',x_D) = -\frac{1}{4} \left[ (x' + 1)^2 + (x_D - 1)^2 - \frac{4}{3} \right] \quad 1 \leq x' < x_D \]  \tag{11}

and

\[ N(x',x_D) = -\frac{1}{4} \left[ (x' - 1)^2 + (x_D + 1)^2 - \frac{4}{3} \right] \quad x_D < x' \leq 1 \]  \tag{12}

3.2 Reservoir Model. Ozkan [8] solved point source problems in Laplace space, further application of point source functions were strengthened. Based on his ideas and methods, pressure distribution formula of vertical fractured wells in circular reservoir can be written as (see Appendix B)

\[ \bar{p}_{wD}(x_D,0,s) = \frac{1}{2} \int_{-1}^{1} \bar{q}_{wD}(x',s)K_0 \left( \frac{[(x_D - x')^2]^{1/2}}{\sqrt{s}} \right) dx' \]  \tag{13}

If the fracture is surrounded by a skin damaged zone, it is easy to know that the total pressure drop of this fracture is equal to the sum value of normal pressure drop in the reservoir and the additional pressure drop caused by the damaged zone, that is

\[ \bar{p}_{wD}(x_D) = \bar{p}_{wD}(x_D,y_D = 0,s) + \bar{q}_{wD}(x_D,s)S_k \]  \tag{14}

Substituting Eqs. (10) and (13) into Eq. (14) will yield

\[ \frac{1}{2} \int_{-1}^{1} \bar{q}_{wD}(x',s)K_0 \left( \frac{[(x_D - x')^2]^{1/2}}{\sqrt{s}} \right) dx' \]

\[ = \left[ \bar{p}_{wD}(s) \right]_{avg} + \frac{\pi}{C_{ID}} \int_{x_D}^{x} N(x',x_D) \bar{q}_{wD}(x',s) - \frac{2\pi}{C_{ID}} S_k N(\theta, x_D) - S_k \bar{q}_{wD}(x_D,s) \]  \tag{15}

Equation (15) is just a new semi-analytical model for asymmetrically fractured wells in an infinite reservoir including skin effects, and we needs to discrete Eq. (15) to get its solution.
3.3 Constant Rate Solution. Assuming the fracture can be divided into 2N segments, integral of the left side in the Eq. (15) would have the following transformation:

\[
\frac{1}{2} \int \hat{q}_{ID}(x', s) K_0 \left( \left( |x_D - x'| \right)^{1/2} \sqrt{s} \right) dx' = \frac{1}{2} \sum_{i=N+1}^{2N} \hat{q}_{ID} \int_{x_D}^{x_{j+1}} K_0 \left( |x_D - x'| \sqrt{s} \right) dx' + \frac{1}{2} \sum_{j=1}^{N} \hat{q}_{ID} \int_{x_D}^{x_{j-1}} K_0 \left( |x_D + x'| \sqrt{s} \right) dx' \tag{16}
\]

Integral of the right side in the Eq. (15) would be transformed as

\[
\frac{\pi}{C_ID} \int_{-1}^{1} N(x', x_D) \hat{q}_{ID}(x', s) dx' = \frac{\pi}{C_ID} \sum_{i=1}^{2N} \int_{x_D}^{x_{j+1}} N(x', x_D) dx' \tag{17}
\]

where \(x_D\) is the midpoint of the \(j\) segment, in addition to the above expressions, by virtue of steady flow, we can know

\[
\frac{1}{2} s \Delta x \sum_{i=1}^{2N} \hat{q}_{ID}(s) = \frac{1}{s} \tag{18}
\]

The unknowns \(\hat{q}_{ID}(s)\) and \(|\hat{p}_{ID}(s)|_{avg}\) can be obtained through combining Eqs. (16)–(18). Then, take \(\hat{q}_{ID}(s)\) and \(|\hat{p}_{ID}(s)|_{avg}\) back into Eq. (10) and let \(x_0 = \theta\) to get the pressure solution in Laplace domain, that is \(\hat{p}_{ID}(\theta, s)\), \(q_{ID}(t_0)\) and \(P_{ID}(t_0)\) for any time given \(t_0\) can be figured out by Stehfest numerical algorithm [30]. To obtain the solution considering the wellbore storage effect, the following relationship must be given as

\[
\hat{P}_{ID} = \frac{1}{s^2 C_D + 1/|\hat{p}_{ID}(\theta, s)\rangle} \tag{19}
\]

If substituting the solved \(\hat{p}_{ID}(\theta, s)\) into Eq. (19), we could obtain the solution with the wellbore storage effect.

3.4 Validation of the Solution. Berumen [27] developed a numerical solution under the constant rate condition. We calculated in the cases of \(\theta = 0.4, S_C = 0, C_D = 10^{-3}, C_{ID} = 0.5\), and \(C_{ID} = 50\) by means of the methods in this paper under the constant rate condition, and it is convenient to compare with the results in the literature (Fig. 2). We can see from Fig. 2 that our results obtained in this paper are consistent with other results in the literature, which will verify the results of this paper should be correct.

4 Results and Case Example

4.1. The Effect of Asymmetry Factor on Flux Distribution. Figures 3 and 4 show that the effects of different asymmetry factors on the flux distribution of the fracture are obvious. Both figures indicate that near the well, the flux will be reaching a peak, and the higher the conductivity is, the lower the peak is and the more uniform the flux distribution is by comparing Fig. 3 with Fig. 4. We can also find that the flux distribution becomes more ununiform and asymmetric as the position of well moved far away from the center of the fracture. For a symmetric fracture (\(\theta = 0\)), flux distribution of two wings along the fracture is uniform; however, for asymmetric fractures, flux of the long wing of the fracture is more lower than the short one and the flux distribution is seriously ununiform when \(\theta = 1\). Therefore, position of the well in the fracture will be an important consideration for effecting on the flux distribution of the fracture.

4.2 The Effect of Asymmetry Factor on Well Test Curves. From Figs. 5 and 6, we can find that in early flow region, under wellbore storage effects, curves of two groups are normalized respectively. While in the later time, the values of both \(P_D\) and \(dP_D\) increase with the increase of wellbore storage coefficient \(C_D\), which means a bigger \(C_D\) will lead to bigger pressure depletion. Contrast Fig. 5 with Fig. 6, we note that the time of wellbore storage effects in Fig. 6 is longer than those in Fig. 5, which means a
bigger fracture asymmetry factor will prolong the time of wellbore storage effect. From Figs. 7 and 8, we can also find that in early flow region, under wellbore storage effects, curves of both groups are normalized respectively. While in the later time, the values of $P_D$ and $dP_D$ both increase with the increase of wellbore storage $C_D$, which means a bigger $C_D$ value will lead to bigger pressure depletion. However, Contrast Fig. 5 with Fig. 7 or contrast Fig. 6 with Fig. 8, we can see that low fracture conductivity will also prolong the time of wellbore storage effects. Therefore, to reduce wellbore storage effects, it is important to keep bigger fracture conductivity and fracture symmetry during the hydraulic fracturing design.

Figures 9 and 10 show the effect of fracture asymmetry factor on type curves ($C_D = 10^{-4}$, $C_D = 10$, $S_k = 0$)
straight line and all curves reach good agreement and the effect of fracture asymmetry factor is not obvious. While in the region of bilinear flow and linear flow, the curves with fracture symmetry are lower than the ones with fracture asymmetry, which shows the existence of the fracture asymmetry will cause bigger pressure depletion. In the radial flow region, all curves show good agreement again. Besides, by comparing Figs. 9 and 10, we find that the larger the value of \( C_D \), the smaller the difference between curves with fracture symmetry and curves with fracture asymmetry. When \( C_D > 10 \), all curves coincide each other and bilinear flow and linear flow can be not observed. This is because for the case of a high wellbore storage coefficient, the wellbore storage effect is dominant and the time of bilinear and linear flow is very short, so we can’t find bilinear flow region and linear flow region. Therefore, for a low wellbore storage coefficient, the effect of fracture asymmetry on early flow is strong.

Figures 11 and 12 show the effect of fracture asymmetry factor \( \theta \) (\( \theta = 0 \) and 1) on dimensionless pressure \( P_D \) and its derivative \( dP_D \) for the same wellbore storage coefficient \( C_D \) and fracture damage factor \( S_k \) but different fracture conductivity \( C_fD \) values of 1, and 50, respectively. By comparing Figs. 11 and 12, we find that the smaller the value of \( C_D \) is, the longer the time of wellbore storage is and when \( S_k > 5 \), type curves with different fracture asymmetry factor nearly coincide each other, which means the existence of fracture damage will decrease the effect of fracture asymmetry. This also indicates that the effect of fracture asymmetry on type curves with smaller fracture damage is very strong.

4.3 Case Example. Table 1 shows reservoir fluid properties and production data of the chosen field reservoir (See Appendix C). Figure 15 shows the matching result of field data and Economides type curves of well with finite conductivity vertical fractures without considering the fracture asymmetry and Fig. 16

<table>
<thead>
<tr>
<th>Name of parameters</th>
<th>Basic data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated net pay thickness, ( h )</td>
<td>235 ft</td>
</tr>
<tr>
<td>Average porosity, ( \phi )</td>
<td>0.18</td>
</tr>
<tr>
<td>Volume factor, ( B )</td>
<td>1,002 Rb/STB</td>
</tr>
<tr>
<td>Viscosity, ( \mu )</td>
<td>0.934 cp</td>
</tr>
<tr>
<td>Total compressibility, ( c_l )</td>
<td>( 6.53 \times 10^{-3} ) Psi(^{-1} )</td>
</tr>
<tr>
<td>Rate, ( q )</td>
<td>334 STB/D</td>
</tr>
</tbody>
</table>
Fracture half-length, we obtain the best matching of the data when the length of two wings. Table 2 shows the summary of results.

<table>
<thead>
<tr>
<th>Sample calculations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability</td>
</tr>
<tr>
<td>From the pressure matching point relation for permeability, ( k ), given as</td>
</tr>
</tbody>
</table>
| \[
| k = 141.2 \frac{qB\mu [p_{wD}]_{MP}}{h [\Delta p]_{MP}} |
| We can obtain the permeability value, \( k \), for this case |
| \[
| k = 141.2 \frac{(334STB/D)(1.002RB/STB)(0.934cp)}{235ft} \frac{1}{234.1} = 0.8md |
| Fracture half length |
| Solving the time matching point relation for fracture half length, \( x_c \), we obtain |
| \[
| x_i = 0.01624 \sqrt{\frac{k}{\phi \mu c D [(l_D/C_D)_{MP}]} [\Delta t]_{MP}} |
| x_f = 0.01624 \sqrt{\frac{0.8md}{(0.18)(0.934cp)(6.53 \times 10^{-6}psi^{-1})}} \frac{1}{10^{-4}} \frac{1}{1} |
| = 133.97ft |
| Finally, the length of two wings of the fracture can be calculated using the following equations: |
| \[
| L_{f1} = x_i (1 - \theta) \quad L_{f2} = x_i (1 + \theta) |
| For this case, we obtain |
| \[
| L_{f1} = 133.97ft \times (1 - 0.8) = 26.8ft \quad L_{f2} = 133.97ft \times (1 + 0.8) = 241.16ft |
| Table 2 shows comparison between new type curves and Economides type curves, which indicates that the fracture asymmetry can be evaluated through new type curves matching, while Economides type curves can’t be used to make the estimation of fracture asymmetry. Contrast Figs. 15 with 16. It can be found that new type curves reach perfect agreement with field data, which implies fracture asymmetry is an important factor in type curves matching. |

5 Conclusions

In this paper, a new constant rate solution for asymmetrically fractured wells is presented. Based on the new solutions, several important conclusions are obtained below.

1. The semi-analytical solutions in this paper are presented in Laplace space, which doesn’t require time-discrete, therefore reducing the account of computation and improving the computational efficiency.

2. As both wellbore storage coefficient and fracture damage factor are considered in the new model, it is convenient to make well test interpretation.

3. New type curves are established for different fracture asymmetry factors and different fracture conductivity. It is shown that a bigger fracture asymmetry factor will prolong the time of wellbore storage effect and low fracture conductivity will also prolong the time of wellbore storage effect. Therefore, to reduce wellbore storage effects, it is important to keep bigger fracture conductivity and fracture symmetric during the hydraulic fracturing design.

4. When the wellbore storage coefficient is small, the effect of fracture asymmetry on early flow is very strong. The existence of the fracture asymmetry will cause bigger pressure loads, which can lead to pressure drop and alter the well test interpretation results.

<table>
<thead>
<tr>
<th>Table 2 Summary of calculative results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Name of parameters</td>
</tr>
<tr>
<td>The conductivity factor, ( C_{D} )</td>
</tr>
<tr>
<td>Storage coefficient, ( C_D )</td>
</tr>
<tr>
<td>Asymmetry factor, ( \theta )</td>
</tr>
<tr>
<td>Fracture half-length, ( x_i )</td>
</tr>
<tr>
<td>Reservoir permeability, ( k )</td>
</tr>
<tr>
<td>Left wing value, ( L_{f1} )</td>
</tr>
<tr>
<td>Right wing value, ( L_{f2} )</td>
</tr>
</tbody>
</table>

shows the matching result of field data and new type curves considering the fracture asymmetry presented in this paper. Using Economides type curves, we get the best matching of the data when \( C_{D} = 5 \) and \( C_{D} = 10^{-4} \) and using new type curves, we obtain the best matching of the data when \( C_{D} = 5 \), \( C_{D} = 10^{-4} \), and \( \theta = 0.8 \). Using the obtained matching points, we then calculate estimates of effective permeability, fracture half-length, and the length of two wings. Table 2 shows the summary of results.

Matching results for new type curves: \( C_{D} = 5 \), \( C_{D} = 10^{-4} \), and \( \theta = 0.8 \)

\[
[t_{0}/C_D]_{MP} = 1 \quad [p_{wD}]_{MP} = 1
\]

\[
[\Delta t]_{MP} = 0.934hr \quad [\Delta p]_{MP} = 243.1psi
\]
depletion and make the starting time of linear flow occur earlier. The effect of fracture asymmetry on the fracture of low conductivity is also very strong. The existence of fracture damage will decrease the effect of fracture asymmetry. This also indicates that the effect of fracture asymmetry on type curves without fracture damage is very strong.

(5) The proposed type curves in this paper can be used to evaluate the fracture asymmetry.

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Nomenclature

Dimensionless Variables: Real Domain

\(a_s\) = asymmetric factor of the fracture in the literature
\(C_{ID}\) = dimensionless fracture conductivity
\(p_{wBD}\) = dimensionless well bottom pressure
\(p_{BD}\) = dimensionless fracture pressure
\(t_D\) = dimensionless time
\(x_{BD}\) = midpoint of the \(j\) segment
\(\theta\) = fracture asymmetry factor

Dimensionless Variables: Laplace Domain

\(S\) = time variable in Laplace domain, dimensionless
\(\tilde{p}_{BD}\) = the pressure \(p_{BD}\) in Laplace domain
\(\tilde{p}_{wBD}\) = bottom pressure \(p_{wBD}\) in Laplace domain
\(\tilde{p}_{BD}\) = fracture pressure \(p_{BD}\) in Laplace domain
\(\tilde{q}(u)\) = fracture rate \(q(x,t)\) in Laplace domain
\(\tilde{q}_{BD}\) = the fracture rate \(q_{BD}\) in Laplace domain

Field Variables

\(c_1\) = total compressibility, \(1/\text{psi}\)
\(k\) = effective permeability, \(\text{md}\)
\(p\) = formation pressure, \(\text{psi}\)
\(p_i\) = initial formation pressure, \(\text{psi}\)
\(\Delta p\) = pressure drop for oil, \(\text{psi}\)
\(q\) = rate of per unit fracture length from formation, \(\text{m}^3/\text{d}\)
\(\mu\) = fluid viscosity, \(\text{cp}\)
\(h\) = formation thickness, \(\text{ft}\)
\(\phi\) = porosity, fraction
\(r\) = reservoir radius, \(\text{ft}\)
\(t\) = time variable, \(\text{h}\)
\(x_t\) = fracture half length, \(\text{ft}\)
\(L_{l1}\) = short wing length of the fracture, \(\text{ft}\)
\(L_{l2}\) = long wing length of the fracture, \(\text{ft}\)
\(w\) = width of the damaged zone, \(\text{ft}\)
\(\lambda\) = integral variable
\(k_{fW}\) = the fracture conductivity, \(\text{md}\ \text{ft}\)

Special Functions

\(K_0(x)\) = modified Bessel function (second kind, zero order)
\(K_1(x)\) = modified Bessel function (second kind, first order)
\(I_0(x)\) = modified Bessel function (first kind, zero order)
\(I_1(x)\) = modified Bessel function (first kind, first order)

Special Subscripts

\(\Omega_1\) = left wing of the fracture
\(\Omega_2\) = right wing of the fracture
\(D\) = dimensionless
\(o\) = oil

Appendix A

First, we must make Laplace transform to Eqs. (6)–(8), the following equations can be obtained:

\[
\frac{\partial^2 \tilde{p}_{BD}}{\partial x_D^2} - \frac{\pi}{C_{ID}} \tilde{q}_{BD} + \frac{2\pi}{x_{BD}} \delta(x_D - \theta) = 0
\]  

(A1)

Outer boundary conditions are given as in Laplace domain

\[
\left(\frac{\partial \tilde{p}_{BD}}{\partial x_D}\right)_{x_D = 0} = 0
\]  

(A2)

\[
\left(\frac{\partial \tilde{p}_{BD}}{\partial x_D}\right)_{x_D = L_{Dl}} = 0
\]  

(A3)

Now, we use Green Function \(N(x', x_D)\) to integrate Eq. (A1), we can know

\[
\int_{x_D}^t N(x', x_D) \left[\frac{\partial \tilde{p}_{BD}}{\partial x'} - \frac{\pi}{C_{ID}} \tilde{q}_{BD}(x', \theta) + \frac{2\pi}{x_{BD}} \delta(x' - \theta)\right] \, dx' = 0
\]  

(A4)

By integrating Eq. (4), it is easy to find the solution, that is

\[
p_{BD}(-1)N(-1, x_D) - p_{BD}N(1, x_D) - \frac{\pi}{C_{ID}} \int_{x_D}^t \tilde{p}_{BD}(x', x_D) \tilde{q}_{BD}(x', s) \, dx' + \frac{2\pi}{C_{ID}x_{BD}} N(\theta, x_D) + \int_{x_D}^t \tilde{p}_{BD}(x', s)N'' \, dx'
\]  

(A5)

and hence require of \(N\) that

\[
N'' = \delta(x' - x_D) + F
\]

\[
F = -\frac{1}{2}
\]  

(A6)

\[N(-1, x_D) = 0
\]

\[N(1, x_D) = 0
\]

Where \(F\) is the correct number, and it is easy to find the solution of Eq. (A6)

\[
N(x', x_D) = \frac{x'^2}{4} + \frac{\lambda x'}{2} \left[Ax' + B(-1 \leq x' < x_D)\right]
\]

(A7)

According to Eq. (A7), we find that continuity of \(N\) at \(x' = x_D\), together with boundary conditions, the unknown coefficients can be given as

\[
\begin{aligned}
A &= -\frac{1}{2}
B &= -\frac{x_D^2}{4} + \frac{x_D}{2} - \frac{1}{6}
C &= \frac{1}{2}
D &= -\frac{x_D^2}{4} - \frac{x_D}{2} - \frac{1}{6}
\end{aligned}
\]

(A8)

Substituting Eq. (A6) into Eq. (A5), the solution can be given as

\[
\tilde{p}_{BD}(x_D, s) = \frac{1}{2} \int_{x_D}^t \tilde{p}_{BD}(x', s) \, dx' + \frac{\pi}{C_{ID}} \int_{x_D}^t N(x', x_D) \tilde{q}_{BD}(x', s) \, dx' - \frac{2\pi}{C_{ID}x_{BD}} N(\theta, x_D)
\]

(A9)
we define

\[
[\tilde{p}_D(s)]_{avg} = \frac{1}{2} \int_1^{\infty} \tilde{p}_D(x', s) dx'
\]  

(A10)

Therefore, substituting Eqs. (A7) and (A8) into Eq. (A9), we can obtain the solution for the fracture model.

**Appendix B**

We consider formation flow model as a plane source in an infinite reservoir, so point source integral method must be used in Laplace domain. The mathematical model of point source for formation flow can be described in Laplace space

\[
\frac{\partial^2 p_D}{\partial x_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \tilde{q}_D
\]  

(B1)

Inner boundary condition of point source can be decided by a changing flow rate, \(q_D(x, t)\), imposed to the wellbore. It can be given as in Laplace domain

\[
- \frac{r_D \partial \tilde{p}_D}{\partial r_D|_{r_D=0}} = \tilde{q}_D(s)
\]  

(B2)

Outer boundary condition is

\[
\tilde{p}_D(\infty, s) = 0
\]  

(B3)

So, the general solution of Eqs. (B1)-(B3) could be expressed as in the form of Bessel Functions

\[
\tilde{p}_D = \tilde{q}_D(s) K_0(r_D \sqrt{s})
\]  

(B4)

Using point sink integral method [8], a single fracture solution can be expressed as

\[
\tilde{p}_D(x_D,0,s) = \frac{1}{2} \int_1^{\infty} \tilde{q}_D(x', s) K_0 \left\{ \left( x_D - x' \right)^2 / \sqrt{s} \right\} dx'
\]  

(B5)

**Appendix C**

SI Metric Conversion Factors

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**References**


