An analytical study on stagnation points in nested flow systems in basins with depth-decaying hydraulic conductivity

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[1] The existence of stagnation points in nested flow systems is relevant to a range of geologic processes. There has been no analytical study on the characteristics and locations of stagnation points in nested flow systems. We derived analytical solutions for hydraulic head and stream function in basins with isotropic and depth-decaying hydraulic conductivity. The solutions of hydraulic head and stream function are used to identify the positions of stagnation points and discuss the dynamics of groundwater around the stagnation points. Three types of stagnation points are identified by analytical and graphical means. For stagnation points on the basin bottom below the valley, only two regional flow systems converge from opposite directions. For stagnation points on the basin bottom below the regional high, only two regional flow systems part toward opposite directions. In contrast, for stagnation points under counterdirectional local flow systems, flow systems converging from and parting toward opposite directions coexist, and these stagnation points move deeper as the water table configuration becomes more rugged and the decay exponent of hydraulic conductivity increases. Moreover, the dividing streamlines around stagnation points under counterdirectional local flow systems are used to divide the local, intermediate, and regional flow systems accurately, from which the penetration depths of local and intermediate flow systems are precisely determined. A clear understanding of the location of stagnation points is critical for characterizing the pattern of hierarchically nested flow systems and has potential implication in studying solute and mineral concentration distributions in drainage basins.


1. Introduction

[2] The theory of regional groundwater flow developed by Tóth [1962, 1963] can be used to explain the contribution of groundwater to many geologic processes [Cardenas, 2007; Garven, 1995; Gleeson and Manning, 2008; Ingebritsen et al., 2006] and has fundamentally affected the subsequent evolution of hydrogeology [Tóth, 2009]. Owing to periodic undulations in the water table, hierarchically nested flow systems, i.e., local, intermediate, and regional, could develop in basins where groundwater is driven by gravity. One important feature of the nested flow pattern is the existence of stagnant zones with extremely low or zero velocity, where metallic ions, hydrocarbons, heat, and anthropogenic contaminants transported by groundwater might accumulate [Tóth, 1999]. Dynamics of groundwater in stagnant zones is critical for understanding these geologic processes.

[3] Mathematically, stagnant zones are associated with stagnation points, where groundwater velocity is zero. It is hardly possible to examine groundwater flow in stagnant zones without identifying the position of stagnation points. Previous investigators [e.g., Anderson and Munter, 1981; Nield et al., 1994; Winter, 1976, 1978] used numerical methods to show that the locations of stagnation points are critical for understanding flow regimes in areas of surface-groundwater interaction. Winter [1976] also discussed the influencing factors of stagnation points numerically. However, numerical solutions are limited in providing insight to the general behavior of groundwater flow. Analytical solutions, on the other hand, are superior in this regard. Unfortunately, no analytical study has been published on stagnation points in nested flow systems. Therefore, analytical analyses of locations and influencing factors of stagnation points are expected to enhance the practical utilization of stagnation points and stagnant zones in a great number of earth science problems such as groundwater age dating, prospecting for ore, petroleum, and geothermal energy, radioactive waste isolation, remediation of polluted water, and lake hydrology.

[4] Recently, Jiang et al. [2009a] found through numerical simulation that the pattern of local versus regional flow is sensitive to the decay exponent, which dictates the rate of decrease in hydraulic conductivity with depth, a widely observed phenomenon in geologic media throughout the world [Jiang et al., 2009b, 2010a; Manning and Ingebritsen, 1999; Neuzil, 1986; Saar and Manga, 2004; Wang et al., 2009]. The distribution of groundwater age in basins is highly correlated with the pattern of local versus
depict groundwater flow systems unambiguously. In turn, important characteristics of flow systems such as penetration depth and stagnation points can be identified.

At Freeze and Witherspoon [1967] derived the analytical solution of groundwater flow paths in basins with depth-decaying hydraulic conductivity. The analytical solutions of hydraulic head and stream function are then used to discuss the characteristics and influencing factors of stagnation points. Finally, the dividing streamlines around stagnation points are used to delineate flow systems and determine penetration depth of local and intermediate flow systems.

2. Mathematical Development

2.1. Mathematical Model

At Freeze and Witherspoon [1967] derived the analytical solution of hydraulic head in homogeneous and isotropic basins with periodic undulation of the water table and gave the definition of flow system. The development of local versus regional flow systems was found to be sensitive to basin geometry [Tóth, 1963] and heterogeneity of hydraulic conductivity [Freeze and Witherspoon, 1967]. Zijl [1999] extended several aspects of the theory of regional groundwater flow by mathematical analysis, proposed a method for calculating penetration depth on the basis of the decay in flux intensity with depth, and suggested that penetration depths can be uniquely determined by the wavelength of water table configuration in isotropic media. In contrast, a recent numerical study by Jianget al. [2009a] concluded that penetration depths increase with the decay exponent of hydraulic conductivity. None of these studies used analytical methods to obtain the streamlines that are essential for division of flow systems. We will determine analytical solutions for the stream function in order to better illustrate groundwater flow paths in basins with depth-decaying hydraulic conductivity and to gain general physical insights into the flows. Streamlines from contouring stream functions

Figure 1. A schematic representation of the cross section of a basin: (a) basin geometry and (b) the two cosine components of the water table configuration.
flow in isotropic media with depth-decaying hydraulic conductivity:
\[
\frac{\partial}{\partial x} \left( K_0 \exp(Az) \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial z} \left( K_0 \exp(Az) \frac{\partial h}{\partial z} \right) = 0. \tag{4}
\]

The boundary conditions can be written as
\[
h(x, 0) = h_0(x), \quad 0 \leq x \leq L; \tag{5a}
\]
\[
\frac{\partial h}{\partial x}_{|z=D} = 0, \quad 0 \leq x \leq L; \tag{5b}
\]
\[
\frac{\partial h}{\partial x}_{|z=0} = 0, \quad D < z < 0; \tag{5c}
\]
\[
\frac{\partial h}{\partial x}_{|z=-L} = 0, \quad D < z < 0. \tag{5d}
\]

[10] Derivation of the analytical solution of equation (4) under the boundary conditions of (5a–5d) is presented in Appendix A. The solution of hydraulic head, \( h \), is (A20)
\[
h(x, z) = (H_R + H_L) + H_R \cos\left( \frac{\pi}{L} x \right) \exp\left( -\frac{A}{2} z \right) - \frac{A \cdot \sinh[a_R(z - D)] + 2a_R \cdot \cosh[a_R(z - D)]}{A \cdot \sinh(a_RD) - 2a_R \cdot \cosh(a_RD)}
\]
\[
+ H_L \cos\left( \frac{\pi}{L} x \right) \exp\left( -\frac{A}{2} z \right) \quad L
\]
\]
\[
\frac{A \cdot \sinh[a_L(z - D)] + 2a_L \cdot \cosh[a_L(z - D)]}{A \cdot \sinh(a_LD) - 2a_L \cdot \cosh(a_LD)}, \tag{6}
\]

where \( a_R = \sqrt{\frac{A^2}{4} + \left( \frac{\pi}{L} \right)^2} \) and \( a_L = \sqrt{\frac{A^2}{4} + \left( \frac{m\pi}{L} \right)^2} \).

2.3. Solution of Stream Function

[11] Streamlines are fundamental to define flow systems. A scalar function whose contour lines define the streamlines is known as the stream function. The stream function, \( \psi(x, z) \), is defined as [Bear, 1972]
\[
\frac{\partial \psi}{\partial x} = -V_x = K_0 \frac{\partial h}{\partial x}, \quad \frac{\partial \psi}{\partial z} = V_z = -K_0 \frac{\partial h}{\partial z}, \tag{7}
\]

where \( V_x \) and \( V_z \) are the horizontal and vertical components, respectively, of Darcy velocity.

[12] For heterogeneous and isotropic media, the stream function for two-dimensional steady-state flow is expressed as
\[
\frac{\partial}{\partial x} \left( \frac{1}{K(x, z)} \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{1}{K(x, z)} \frac{\partial \psi}{\partial z} \right) = 0. \tag{8}
\]

Applying equation (3) to (8), we obtain the equation for two-dimensional steady-state groundwater flow in isotropic media with depth-decaying hydraulic conductivity:
\[
\frac{\partial}{\partial x} \left( K_0 \exp(Az) \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial z} \left( K_0 \exp(Az) \frac{\partial \psi}{\partial z} \right) = 0. \tag{9}
\]

The boundary conditions can be written as
\[
\psi(x, 0) = \psi_0(x), \quad 0 \leq x \leq L; \tag{10a}
\]
\[
\psi(x, D) = 0, \quad 0 \leq x \leq L; \tag{10b}
\]
\[
\psi(0, z) = 0, \quad D < z < 0; \tag{10c}
\]
\[
\psi(L, z) = 0, \quad D < z < 0; \tag{10d}
\]

where \( \psi_0(x) \) can be calculated as
\[
\psi_0(x) = \int_0^x V_z(x, 0) \, dx = -K_0 \int_0^x \frac{\partial h}{\partial z} \, dz. \tag{11}
\]

According to equation (6), the derivative of \( h \) with respect to \( z \) is
\[
\frac{\partial h}{\partial z} = H_R \cos\left( \frac{\pi}{L} x \right) \exp\left( -\frac{A}{2} z \right) \frac{2a_R^2 - A^2}{A \cdot \sinh(a_RD) - 2a_R \cdot \cosh(a_RD)}
\]
\[
+ H_L \cos\left( \frac{m\pi}{L} x \right) \exp\left( -\frac{A}{2} z \right) \frac{2a_L^2 - A^2}{A \cdot \sinh(a_LD) - 2a_L \cdot \cosh(a_LD)}. \tag{12}
\]

Applying equation (12) to equation (11), we obtain
\[
\psi_0(x) = -\frac{K_0H_R}{\pi} \sin\left( \frac{\pi}{L} x \right) - \frac{K_0H_L}{m\pi} \sin\left( \frac{m\pi}{L} x \right), \tag{13}
\]

where \( A_R \) and \( A_L \) are constants dependent on \( D \) and \( A \). The expressions for \( A_R \) and \( A_L \) are
\[
A_R = \frac{\left( \frac{A^2}{2} - 2\pi^2 \right) \cdot \sinh(a_RD)}{A \cdot \sinh(a_RD) - 2a_R \cdot \cosh(a_RD)}, \quad \text{and}
\]
\[
A_L = \frac{\left( \frac{A^2}{2} - 2\pi^2 \right) \cdot \sinh(a_LD)}{A \cdot \sinh(a_LD) - 2a_L \cdot \cosh(a_LD)}. \tag{14}
\]

[14] Derivation of the analytical solution to equation (9) for stream function under the boundary conditions of (10a–10d) is presented in Appendix B. The solution of stream function is (B19):
\[
\psi(x, z) = \frac{K_0L}{\pi} H_R A_R \sin\left( \frac{\pi}{L} x \right) \exp\left( \frac{A}{2} z \right) \frac{\sinh[a_R(z - D)]}{\sinh(a_RD)}
\]
\[
+ \frac{K_0L}{m\pi} H_L A_L \sin\left( \frac{m\pi}{L} x \right) \exp\left( \frac{A}{2} z \right) \frac{\sinh[a_L(z - D)]}{\sinh(a_LD)}. \tag{14}
\]

We define the first part of the right side of equation (14) as \( \psi_R \), representing the component in the stream function contributed by the regional relief of the water table, and define the second part as \( \psi_L \), representing stream function contributed by the local relief.
2.4. Stagnation Points

According to Bear [1972], the stagnation points are defined by

\[ V_x = 0 \quad \text{and} \quad V_z = 0. \]  \hspace{1cm} (15)

The Darcy velocity can be calculated as

\[ V_x = K_0 H_R \frac{\pi}{L} \sin \left( \frac{\pi}{L} x \right) \exp \left( \frac{A}{2} \right) \]
\[ \cdot \left( \frac{A \cdot \sinh[a_k(z - D)] + 2a_R \cdot \cosh[a_k(z - D)]}{A \cdot \sinh[a_k D] - 2a_R \cdot \cosh[a_k D]} \right) \]
\[ + K_0 H_L \frac{\pi}{L} \sin \left( \frac{\pi}{L} x \right) \exp \left( \frac{A}{2} \right) \]
\[ \cdot \frac{A \cdot \sinh[a_L(z - D)] + 2a_L \cdot \cosh[a_L(z - D)]}{A \cdot \sinh[a_L D] - 2a_L \cdot \cosh[a_L D]} ; \]  \hspace{1cm} (16a)

\[ V_z = -K_0 H_R \cos \left( \frac{\pi}{L} x \right) \exp \left( \frac{A}{2} \right) \]
\[ \cdot \left( \frac{2a_R^2 - A^2}{A} \right) \cdot \frac{\sinh[a_k(z - D)]}{A \cdot \sinh[a_k D] - 2a_R \cdot \cosh[a_k D]} \]
\[ -K_0 H_L \cos \left( \frac{\pi}{L} x \right) \exp \left( \frac{A}{2} \right) \]
\[ \cdot \left( \frac{2a_L^2 - A^2}{A} \right) \cdot \frac{\sinh[a_L(z - D)]}{A \cdot \sinh[a_L D] - 2a_L \cdot \cosh[a_L D]} . \]  \hspace{1cm} (16b)

The stagnation points can be located by plotting the distribution of \( V_x \) and \( V_z \) on the same figure. The points where both \( V_x \) and \( V_z \) equal 0 can be accepted as stagnation points. Alternatively, stagnation points can be determined mathematically by combining equations (16a), (16b), and (15). We may find that both \( V_x \) and \( V_z \) are equal to 0 at \( x = 0 \) and \( x = L \) on the basin bottom. At \( x = L/2 \), \( V_z \) is absolutely and \( V_x \) is possibly 0. Therefore, there are stagnation points at \( x = 0 \) and \( x = L \) and might be a stagnation point at \( x = L/2 \). More specifically, at \( x = L/2 \), if there is a counterdirectional local flow system, then a stagnation point exists; in contrast, if there is a codirectional local flow system, then no stagnation point exists (counterdirectional and codirectional local flow systems will be defined in section 3.2). In other parts of the basin, the stagnation points can be found by simultaneously solving the following equations obtained by setting equations (16a) and (16b) to 0:

\[ A \cdot \sinh[a_k(z - D)] + 2a_R \cdot \cosh[a_k(z - D)] \]
\[ \cdot \frac{A \cdot \sinh[a_k D] - 2a_R \cdot \cosh[a_k D]}{A \cdot \sinh[a_L(z - D)] + 2a_L \cdot \cosh[a_L(z - D)]} \]
\[ = -\frac{H_L a_k - A^2}{H_R A \cdot \sinh[a_k D] - 2a_R \cdot \cosh[a_k D]} \sin \left( \frac{\pi}{L} x \right) \sin \left( \frac{\pi}{L} z \right); \]  \hspace{1cm} (17a)

\[ \frac{\sinh[a_k(z - D)]}{\sinh[a_L(z - D)]} = -\frac{H_L}{H_R} \frac{4a_R^2 - A^2}{4a_L^2 - A^2} \]
\[ \cdot \frac{A \cdot \sinh[a_k D] - 2a_R \cdot \cosh[a_k D]}{A \cdot \sinh[a_L D] - 2a_L \cdot \cosh[a_L D]} \cos \left( \frac{\pi}{L} x \right) \cos \left( \frac{\pi}{L} z \right). \]  \hspace{1cm} (17b)

Equations (17a) and (17b) show that the positions of stagnation points (except for the two at \( x = 0 \) and \( x = L \)) are dependent on the wavelengths of local and regional undulations, the amplitudes of local and regional undulations, the decay exponent, and the depth of the basin. In the following, we use the analytical solutions described above to discuss the flow pattern around stagnation points, the influencing factors, and utilization of stagnation points.

3. Characteristics of Stagnation Points

3.1. Simplified Solution for Homogeneous and Isotropic Case (\( A = 0 \))

For clarity of analysis, we use \( A = 0 \), representing a homogeneous and isotropic basin, as a special case to analyze the characteristics of stagnation points. Substituting \( A = 0 \) into equations (6) and (14), we obtain more concise equations of the hydraulic head and stream function:

\[ h(x,z) \big|_{A=0} = (H_R + H_L) - H_R \cos \left( \frac{\pi}{L} z \right) \cosh \left( \frac{\pi}{L} D \right) \]
\[ - H_L \cos \left( \frac{\pi}{L} x \right) \cosh \left( \frac{\pi}{L} D \right) \cosh \left( \frac{m\pi}{L} D \right) \cosh \left( \frac{m\pi}{L} D \right); \]  \hspace{1cm} (18)

\[ \psi(x,z) \big|_{A=0} = K_0 H_R \sin \left( \frac{\pi}{L} x \right) \frac{\sin \left( \frac{\pi}{L} z \right)}{cosh \left( \frac{\pi}{L} D \right)} \]
\[ + K_0 H_L \sin \left( \frac{m\pi}{L} x \right) \frac{\sin \left( \frac{m\pi}{L} z \right)}{cosh \left( \frac{m\pi}{L} D \right)}. \]  \hspace{1cm} (19)

Similarly, the Darcy velocity can be expressed as

\[ V_x \big|_{A=0} = -K_0 \frac{\pi}{L} \left[ H_R \sin \left( \frac{\pi}{L} x \right) \cos \left( \frac{\pi}{L} z \right) \right. \]
\[ + mH_L \sin \left( \frac{m\pi}{L} x \right) \frac{\sin \left( \frac{m\pi}{L} z \right)}{cosh \left( \frac{m\pi}{L} D \right)} \left. \right]; \]  \hspace{1cm} (20a)

\[ V_z \big|_{A=0} = K_0 \frac{\pi}{L} \left[ H_R \cos \left( \frac{\pi}{L} x \right) \sin \left( \frac{\pi}{L} z \right) \right. \]
\[ + mH_L \cos \left( \frac{m\pi}{L} x \right) \frac{\sin \left( \frac{m\pi}{L} z \right)}{cosh \left( \frac{m\pi}{L} D \right)} \left. \right]. \]  \hspace{1cm} (20b)

3.2. Flow Fields Illustrated by Hydraulic Head and Streamlines

We compute hydraulic head and stream function using equations (18) and (19). The parameters are chosen to ascertain that all orders of flow systems, i.e., local, intermediate, and regional, are developed. The width of the basin, \( L \), is 7000 m and the elevation of the basin bottom is
The wavelength of the regional variation of water table is 14,000 m while the wavelength of the local relief of water table is 2000 m, which leads to the value of \( m \) to be 7. The amplitudes of both regional and local undulations are assumed to be 50 m. The hydraulic conductivity at the ground surface is assumed to be 1 m/d.

The flow field is shown in Figure 2a, with dashed black lines representing contours of hydraulic head, red lines representing contours of stream function, and blue arrows representing flow directions. All orders of flow systems are well developed. Owing to the periodic undulations of the water table, there are seven local flow systems (L1 through L7), one intermediate flow system (I), and one regional flow system (R). Among the seven local flow systems, the flow direction in L1, L3, L5, and L7 is the same as that in the higher-order flow systems below them, while the flow direction in L2, L4, and L6 is opposite to that in the higher-order flow systems below them. Here, we call the local flow systems with similar flow direction to the regional system as codirectional local flow systems and those with opposite direction to the regional one as counterdirectional local flow systems.

We plot the stream functions contributed by the regional undulation (\( \psi_R \)) and by the local relief (\( \psi_L \)) separately (Figures 2b and 2c). The stream function shown in Figure 2a is the sum of stream functions shown in Figures 2b and 2c. Under the boundary conditions of equations (10a–10d), \( \psi_R \) in Figure 2b has all positive values and increases with

![Figure 2](image_url)

**Figure 2.** Distribution of hydraulic head and stream function for \( A = 0 \). (a) The hydraulic head (dashed black line), stream function (solid red line), and flow direction (blue arrow). (b) Stream function contributed by regional undulation, \( \psi_R \). (c) Stream function contributed by local undulation, \( \psi_L \).
elevation, which indicates that $\partial y_L/\partial z > 0$. $\psi_L$ in Figure 2c has positive values in codirectional local flow systems and negative values in counterdirectional local flow systems. In codirectional local flow systems, $\partial \psi_L/\partial z > 0$, while in counterdirectional local flow systems, $\partial \psi_L/\partial z < 0$. Acting together, $\partial \psi/\partial z$ is positive in codirectional local flow systems, but changes from negative to positive values as $z$ decreases in counterdirectional local flow systems, which implies that $\partial \psi/\partial z = 0$ is within the counterdirectional local flow systems.

3.3. Positions of Stagnation Points

To locate the positions of stagnation points, we show the contours of $V_x$ and $V_z$ in Figure 3a. The thick lines are $V_x = 0$ and $V_z = 0$. It is clear that there are five stagnation points, indicated as SP 1, SP 2, SP 3, SP 4, and SP 5. SP 4 is located on the basin bottom at $x = 0$ and SP 5 is located on the basin bottom at $x = L$. They change neither with amplitudes of local and regional undulations nor with the decay exponent. On the other hand, the positions of SP 1, SP 2, and SP 3 are correlated with the pattern of the nested flow systems, which is highly dependent on such factors as local versus regional undulations, the decay exponent of hydraulic conductivity, and the basin depth. Moreover, we find that SP 1, SP 2, and SP 3 are below the counterdirectional local flow systems, which suggests that the development of stagnation points related to local flow systems is dependent on the existence of counterdirectional local flow systems. The depth of SP 1 is the same as that of SP 3, but differs from that of SP 2. Moreover, the penetration depth of counterdirectional local flow systems is associated with these stagnation points. These findings have implications, for example, in interpreting groundwater age distribution or studying potential sites of concentrating minerals or petroleum, which might exist in stagnant zones.

We also plot the distribution of absolute Darcy velocity, $V$, calculated by $V = \sqrt{V_x^2 + V_z^2}$, in Figure 3b. At the stagnation points, the absolute Darcy velocity equals 0. Around these stagnation points, the absolute Darcy velocity is small enough to form quasi stagnant zones. The boundaries of the shaded zones in Figure 3b are chosen at a value of 0.005 m/d. This value can be set arbitrarily depending on the purpose.

3.4. Groundwater Dynamics Around Stagnation Points

To better illustrate the stagnant zones, we zoom in on the flow patterns around SP 1 and SP 2, as shown in Figure 4. Take the four flow systems around SP 1 for example (Figure 4a). If the intermediate and regional flow systems (I and R) are considered as one pair, and the first and second local flow systems (L1 and L2) as another pair, we find that streamlines flow in the same direction first and then part toward opposite directions. If, however, L2 and I are considered as a pair, and L1 and R as another one,
streamlines converge from opposite directions first and then flow in the same direction. A similar phenomenon can be found in the four flow systems of different order around SP 2 (Figure 2a and Figure 4b) and SP 3 (Figure 2a). We conclude that flow systems converging from and parting toward opposite directions coexist around stagnation points that develop below counterdirectional local flow systems. If, on the other hand, we consider the cross section of the entire basin, only two regional flow systems converge toward the stagnation point (SP 4) below the regional valley, and only two regional flow systems part toward opposite directions at the stagnation point (SP 5) below the regional water divide (Figures 2a and 3a).

Another important feature of flow around stagnation points is a potentiometric minimum \cite{Töth, 1988}. On the basis of this feature, monitoring of hydraulic head in the field can help determine the existence of stagnation points \cite{Anderson and Munter, 1981}. The phenomenon of potentiometric minimum is shown in the contour of hydraulic head in Figure 4. To characterize the potentiometric minimum visually, we use three-dimensional plots to show the distribution of hydraulic head (Figure 5). The shape of a saddle illustrates the condition of the potentiometric minimum around stagnation points. Moreover, the hydraulic head shown in Figure 5 indicates that both $\partial h/\partial x$ and $\partial h/\partial z$ equal 0. In fact, at stagnation points, not only $V_x$, $V_z$, $\partial h/\partial x$, and $\partial h/\partial z$ equal 0, but $\partial w/\partial x$ and $\partial w/\partial z$ also equal 0, as indicated by the definition of stream function in equation (7).

4. Influencing Factors of Stagnation Points

In this section, we only consider the stagnation points under counterdirectional local flow systems. When the basin depth and the wavelength of the water table relief are fixed, the position of these stagnation points are found to be sensitive to the amplitudes of local and regional undulations of water table and the decay exponent. Sensitivity studies show that an increase in amplitude of local relief of water table, $H_L$, a decrease in amplitude of regional undulation, $H_R$, and an increase in decay exponent, $A$, all move the stagnation points toward the basin bottom (Figure 6). The quantitative relationship between the elevation of SP 2 and the decay exponent, $A$, and the ratio of amplitudes of local to regional undulation, $H_L/H_R$, is shown in Figure 7. Note that the curve for $z$ versus $A$ is convex upward, which implies that the rate of deepening increases with $A$, while the curve for $z$ versus $H_L/H_R$ is convex downward, which indicates that the rate of deepening decreases with $H_L/H_R$. On the basis of Figures 6 and 7, we may infer that when the basin is not deep enough, but both $A$ and $H_L/H_R$ are great enough, the stagnation

Figure 4. Distribution of hydraulic head (dashed black lines) and streamlines (solid red lines) around stagnation points (a) SP 1 and (b) SP 2.

Figure 5. Three-dimensional maps showing potentiometric minimum around stagnation points (a) SP 1 and (b) SP 2. The black lines are contours of hydraulic head and the red lines are contours of stream function.
points could reach the basin bottom and only local flow systems could develop. Therefore, in shallow basins, if the topographic variation is drastic and the decay exponent is significant, only localized flow systems are expected and no basin-scale flow occurs. Such situations tend to promote the compartmentalization of extensive surface-water drainage basins into several subbasins of groundwater flow.

The distributions of hydraulic head and stream function under different $H_R$, $H_L$, and $A$ are compared with the base case of $H_R = 50$ m, $H_L = 50$ m, and $A = 0$ (Figure 2a). Figure 8a shows the result for $H_R$ being reduced to 25 m, Figure 8b for $H_L$ being increased to 100 m, and Figure 8c for $A$ being increased to 0.001 m$^{-1}$. A common phenomenon emerging from the three cases is that the contrast between streamline densities in the shallow part and in the deep part is much greater than that of the base case shown in Figure 2a. For a fixed contour interval of 5, the streamlines show that regional flow is absent in Figure 8a, and both intermediate and regional flow systems are absent in Figure 8c. The great contrast of streamline density is a direct result of greater intensity of local versus regional flow, which would certainly lead to deepening of stagnation points.

Note that in the two cases shown in Figures 8a and 8b, $H_L/H_R$ equals 2. Compared with the base case shown in Figure 2a, the topography of the water table in Figure 8a is much "gentler" and the topography of the water table in Figure 8b is not only more rugged but also regionally steeper. The shape of contours of hydraulic head shown in Figures 8a and 8b is the same, but the value of hydraulic head in case (a) is half of that in case (b). Likewise, the shape of contours of stream function in the two cases is also the same, although the density of streamlines in case (b) is higher. Such patterns of hydraulic head and stream function...
distribution would certainly lead to identical distribution of stagnation points.

5. Division of Flow Systems Using Stagnation Points

[29] In a flow system, two streamlines adjacent at one point of the flow region should remain adjacent through the region [Tóth, 1963]. We found in section 3 that flow systems part and meet around the stagnation points under counterdirectional local flow systems and the streamlines around these stagnation points can be arbitrarily close to, but never reach, the stagnation points. This offers a method to divide the flow systems, which is described as follows. After identifying positions of stagnation points based on equation (15), we can calculate the stream function at points arbitrarily close to the stagnation points and trace the corresponding streamlines. We find that for our case, 10 different streamlines around the three stagnation points are enough to divide the flow systems (Figure 9). Such streamlines were defined as dividing streamlines by Nield et al. [1994]. For the local flow systems in the base case, both L1 and L7 are bounded by the dividing streamlines with \( \psi = 10.29 \), both L2 and L6 by \( \psi = 10.27 \), both L3 and L5 by \( \psi = 20.92 \), and L4 by \( \psi = 20.90 \). The intermediate flow system, I, is bounded by two dividing streamlines with \( \psi = 20.90 \) and \( \psi = 10.29 \). The streamline with \( \psi = 10.27 \) constitutes the upper boundary of the regional flow system, R. Note that if we choose values slightly larger than 10.27 and 20.90, and slightly smaller than 10.29 and 20.92, the white zones around the stagnation points would be smaller.

[30] We note in section 3.3 that the penetration depth of counterdirectional local flow systems can be determined by the position of stagnation points under them. On the basis of the division of flow systems, we can also determine the penetration depth of codirectional local flow systems accurately. As shown in Figure 9a, the penetration depth of codirectional local flow systems is larger than that of their twin counterdirectional local flow systems. This is in accordance with our previous finding using numerical simulation that the penetration depths of different local flow systems are different [Jiang et al., 2009a]. Therefore, the location of stagnation points is a useful index to characterize the pattern of topography-driven groundwater flow in drainage basins.

[31] We further examine the divisions of flow systems in a basin with depth-decaying hydraulic conductivity (Figure 10). We found that the size of flow systems is sensitive to the decay exponent, \( A \). As \( A \) increases, the sizes of the regional and intermediate flow systems decrease, and

![Figure 7](image-url)
the sizes of the recharge and discharge areas of the regional and intermediate flow system also decrease. This indicates that increasing \( A \) restricts regional flow and intermediate flow systems, thus allowing the penetration depth of local and intermediate flow systems to increase. Consequently, the stagnation points move toward the basin bottom.

6. Conclusions

[32] We have derived equations of hydraulic head and stream function for drainage basins where groundwater is driven by gravity and hydraulic conductivity is isotropic but decays exponentially with depth. The basin geometry is simplified to a rectangle with the top boundary condition representing the water table as a subdued replica of topography. The equation for the water table configuration is represented by two cosine functions instead of a linear trend and a sine function for mathematical convenience.

[33] The stagnation points, where flow velocities equal 0, are obtained by finding the point of intersection of contours with \( V_x = 0 \) and \( V_z = 0 \) or by deriving the expressions mathematically. There are three kinds of stagnation points. The first is located on the bottom below the principal valley, where two regional flow systems meet in the cross section of a complete basin. The second is located on the basin bottom below the main divide, where two regional flow systems part. The lateral position of these stagnation points is dependent only on the configuration of the basin. The third kind of stagnation points is located below counterdirectional local flow systems where flow systems of different order meet and part simultaneously. By plotting streamlines using contours of stream function, the flow pattern around stagnation points is rigorously defined and clearly illustrated.

[34] Both an increase in the ratio of the amplitudes of local to regional undulations and in decay exponent of hydraulic conductivity with depth lead to a lowering of the

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**Figure 8.** Distributions of hydraulic head and stream function resulting from different conditions: (a) \( H_L = 50 \text{ m}, \ H_R = 25 \text{ m}, \ A = 0 \text{ m}^{-1} \), (b) \( H_L = 100 \text{ m}, \ H_R = 50 \text{ m}, \ A = 0 \text{ m}^{-1} \), and (c) \( H_L = 50 \text{ m}, \ H_R = 50 \text{ m}, \ A = 0.001 \text{ m}^{-1} \).
Figure 9. Division of flow systems based on streamlines around stagnation points: (a) flow systems, (b) values of stream function around SP 1, (c) values of stream function around SP 2, and (d) values of stream function around SP 3.
Figure 10. Development of flow systems under depth-decaying hydraulic conductivity: (a) $A = 0.001 \text{ m}^{-1}$, (b) $A = 0.003 \text{ m}^{-1}$, and (c) $A = 0.005 \text{ m}^{-1}$.
third kind of stagnation points, i.e., those below counter-
directional local flow systems. In both cases, the intensity of
local flow versus regional flow increases. When the basin is
not deep enough, but both decay exponent and ratio of
amplitudes of local to regional undulations are great enough,
the stagnation points may reach the basin bottom, and,
consequently, only local flow systems can develop.

[35] The streamlines around stagnation points can be used
to delineate the flow systems accurately, and thus determine
penetration depth of both codirectional and counter-
directional local flow systems easily. Therefore, the location
of stagnation points is fundamental to characterize the pat-
tern of groundwater flow in topography-driven basins.

[36] Offering a theoretical understanding and character-
ization of the distribution of stagnation points and how they
vary with topography of the water table, this study has
potential applications in studying age distribution of ground-
water, solute and mineral concentrations and petroleum migration in complex topographic basins with depth-decaying
hydraulic conductivity.

Appendix A

[37] Analytical solution to equation (4) for hydraulic head
is presented in following.

[38] Introducing \( u = e^{xL}h \), equation (4) can be transformed to
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} - A \frac{\partial u}{\partial z} = 0.
\]

Let
\[
u(x, z) = X(x) Z(z) .
\]

Applying equation (A2) to equation (A1), we obtain the
following two ordinary differential equations:
\[
\frac{d^2 X}{dx^2} + \lambda X = 0 ,
\]
\[
\frac{d^2 Z}{dz^2} - A \frac{dZ}{dz} - \lambda Z = 0 ,
\]

where \( \lambda \) is constant.

[39] The general solution for equation (A3) is
\[
X(x) = C^* \cos \sqrt{\lambda} x + D^* \sin \sqrt{\lambda} x .
\]

According to equations (5c) and (5d), the boundary con-
tions for \( X(x) \) are
\[
X'(0) = 0 \quad \text{and} \quad X'(L) = 0 .
\]

Therefore, in equation (A5), \( D^* = 0 \) and \( \sqrt{\lambda} = \pi n / L , n = 0, 1, 2, \ldots \), and the characteristic function for \( X(x) \) is
\[
X_n(x) = C_n \cos \left( \frac{n \pi x}{L} \right) , \quad n = 0, 1, 2, \ldots .
\]

[40] The general solution for equation (A4) is
\[
Z(z) = e^{z} \left( C'' \exp \left( \frac{A^2}{4} + \lambda z \right) + D'' \exp \left( - \frac{A^2}{4} + \lambda z \right) \right) .
\]

Considering the numerous values of \( \lambda \), the characteristic
function for \( Z(z) \) is
\[
Z_n(z) = C_n \exp \left( \frac{A^2}{4} + \frac{n \pi z^2}{L^2} \right) + D_n \exp \left( - \frac{A^2}{4} + \frac{n \pi z^2}{L^2} \right) , \quad n = 0, 1, 2, \ldots .
\]

[41] To keep equations concise, we introduce
\( a_n = \sqrt{\frac{A^2}{4} + \left( \frac{n \pi}{L} \right)^2} \). According to equations (A7) and (A9), the solution of \( u \) is
\[
u(x, z) = \sum_{n=0}^{\infty} X_n(x) Z_n(z) = \sum_{n=0}^{\infty} \cos \left( \frac{n \pi x}{L} \right) \left[ C_n \exp \left( \frac{A}{2} + a_n \right) z + D_n \exp \left( \frac{A}{2} - a_n \right) z \right].
\]

The solution of \( h \) can be obtained by applying \( h = u e^{-Az} \):
\[
h(x, z) = \sum_{n=0}^{\infty} \cos \left( \frac{n \pi x}{L} \right) \left[ C_n \exp \left( - \frac{A}{2} + a_n \right) z + D_n \exp \left( - \frac{A}{2} - a_n \right) z \right].
\]

The values of \( C_n \) and \( D_n \) are obtained by applying the
boundary conditions as expressed in equations (5a) and (5b). According to equation (A11), the hydraulic head at \( z = 0 \) is
\[
h(x, 0) = \sum_{n=0}^{\infty} (C_n + D_n) \cos \left( \frac{n \pi x}{L} \right).
\]

[42] Note that equations (1), (5a), and (A12) are equiva-
lent. By using cosine series, for \( n = 0 \),
\[
C_0 + D_0 = \frac{2}{L} \int_0^L \left[ (H_L + H_R) - H_R \cos \left( \frac{n \pi x}{L} \right) - H_L \cos \left( \frac{m \pi x}{L} \right) \right] dx = 2(H_L + H_R).
\]

For \( n = 1, 2, 3, \ldots \),
\[
C_n + D_n = \frac{2}{L} \int_0^L \left[ (H_L + H_R) - H_R \cos \left( \frac{n \pi x}{L} \right) \right] \cos \left( \frac{n \pi x}{L} \right) dx .
\]

From equations (A13) and (A14), we can get
\[
C_n + D_n = \begin{cases} 2(H_L + H_R), & n = 0 \\ -H_R, & n = 1 \\ -H_L, & n = m \\ 0, & \text{others} \end{cases}
\]
By combining equation (5b) and the derivative of \( h \) with respect to \( z \) calculated from equation (A11),

\[
\frac{\partial h}{\partial z} = \sum_{n=0}^{\infty} \cos \left( \frac{n\pi}{L} x \right) \left\{ C_n \left( \frac{A}{2} + a_n \right) \exp \left[ \left( \frac{A}{2} + a_n \right) D \right] + D_n \left( -\frac{A}{2} - a_n \right) \exp \left[ \left( -\frac{A}{2} - a_n \right) D \right] \right\} = 0. \quad (A16)
\]

Therefore,

\[
C_n \left( -\frac{A}{2} + a_n \right) \exp \left[ \left( -\frac{A}{2} + a_n \right) D \right] + D_n \left( -\frac{A}{2} - a_n \right) \exp \left[ \left( -\frac{A}{2} - a_n \right) D \right] = 0. \quad (A17)
\]

By solving equations (A15) and (A17), we can get the values of \( C_n \) and \( D_n \),

\[
C_n = \begin{cases} 
2(H_L + H_R), & n = 0 \\
\left( -\frac{A}{2} + a_1 \right) \exp \left( -a_1 D \right) - A \cdot \sinh(a_1 D) + 2a_1 \cdot \cosh(a_1 D), & n = 1 \\
\frac{-A \cdot \sinh(a_n D) + 2a_1 \cdot \cosh(a_n D)}{\frac{-A}{2} - a_n} \exp \left( -a_n D \right) H_L, & n = m \\
\frac{-A \cdot \sinh(a_m D) + 2a_1 \cdot \cosh(a_m D)}{\frac{-A}{2} - a_m} \exp \left( -a_m D \right) H_L, & n = m \\
0, & \text{others}
\end{cases} \quad (A18)
\]

\[
D_n = \begin{cases} 
0, & n = 0 \\
\left( -\frac{A}{2} + a_1 \right) \exp \left( a_1 D \right) - A \cdot \sinh(a_1 D) + 2a_1 \cdot \cosh(a_1 D) H_R, & n = 1 \\
\frac{-A \cdot \sinh(a_n D) + 2a_1 \cdot \cosh(a_n D)}{\frac{-A}{2} + a_n} \exp \left( a_n D \right) H_R, & n = m \\
\frac{-A \cdot \sinh(a_m D) + 2a_1 \cdot \cosh(a_m D)}{\frac{-A}{2} + a_m} \exp \left( a_m D \right) H_R, & n = m \\
0, & \text{others}
\end{cases} \quad (A19)
\]

Let

\[
w(x, z) = X(x) Z(z). \quad (B2)
\]

Note that \( X(x) \) and \( Z(z) \) in this part are independent of the \( X(x) \) and \( Z(z) \) in Appendix A.

[46] Applying equation (B2) to equation (B1), we obtain the following two ordinary differential equations:

\[
\frac{d^2 X}{dx^2} - \gamma X = 0, \quad (B3)
\]

\[
\frac{d^2 Z}{dz^2} + A \frac{dZ}{dz} - \gamma Z = 0, \quad (B4)
\]

where \( \gamma \) is constant.

[47] The general solution for equation (B3) is

\[
X(x) = P' \cos \sqrt{\gamma} z + Q' \sin \sqrt{\gamma} z. \quad (B5)
\]

According to equations (10c) and (10d), the boundary conditions for \( X(x) \) are

\[
X(0) = 0 \quad \text{and} \quad X(L) = 0. \quad (B6)
\]

Therefore, in equation (B5), \( P' = 0 \) and \( \sqrt{\gamma} = n \pi / L \), \( n = 1, 2, \ldots \), and the characteristic function for \( X(x) \) is

\[
X_n(x) = Q_n' \sin \left( \frac{n \pi}{L} x \right), \quad n = 1, 2, \ldots. \quad (B7)
\]

[48] The general solution for equation (B4) is

\[
Z(z) = e^{\pm i \theta} \left( P_n'' \exp \left( \frac{A^2}{4} + \gamma z \right) + Q_n'' \exp \left( -\frac{A^2}{4} + \gamma z \right) \right). \quad (B8)
\]

Considering the numerous values of \( \gamma \), the characteristic function for \( Z(z) \) is

\[
Z_n(z) = P_n'' \exp \left( \frac{-A^2}{4} + \frac{\left( n \pi \right)^2}{L^2} z \right) + Q_n'' \exp \left( -\frac{A^2}{4} + \frac{\left( n \pi \right)^2}{L^2} z \right), \quad n = 0, 1, 2, \ldots. \quad (B9)
\]

To keep equations concise, we also introduce \( a_n = \sqrt{\frac{A^2}{4} + \frac{\left( n \pi \right)^2}{L^2}} \). According to equations (B7) and (B9), the solution of \( w \) is

\[
w(x, z) = \sum_{n=1}^{\infty} X_n(x) Z_n(z) = \sum_{n=1}^{\infty} \sin \left( \frac{n \pi}{L} x \right) \left \{ P_n \exp \left[ -\frac{A}{2} + a_n \right] z \right \} + Q_n \exp \left[ -\frac{A}{2} - a_n \right] z \right \} \quad (B10)
\]

[49] The solution of \( \psi \) can be obtained by applying \( \psi = \text{we}^{i \psi} \):

\[
\psi(x, z) = \sum_{n=1}^{\infty} \sin \left( \frac{n \pi}{L} x \right) \left \{ P_n \exp \left[ \frac{A}{2} + a_n \right] z \right \} + Q_n \exp \left[ \frac{A}{2} - a_n \right] z \right \}. \quad (B11)
\]
The values of $P_n$ and $Q_m$ are obtained by applying the boundary conditions as expressed in equations (10a) and (10b). According to equation (B11), the stream function at $z = 0$ is
\[
\psi(x, 0) = \sum_{n=1}^{\infty} (P_n + Q_n) \sin \left( \frac{n\pi x}{L} \right). \tag{B12}
\]

[50] Note that equations (10a), (13), and (B12) are equivalent. By using sine series,
\[
P_n + Q_n = \frac{2}{L} \int_0^L \left[ -\frac{K_0 H_A A_L}{\pi} \sin \left( \frac{n\pi x}{L} \right) - \frac{K_0 H_A A_L}{m\pi} \sin \left( \frac{m\pi x}{L} \right) \right] \sin \left( \frac{n\pi x}{L} \right) \, dx. \tag{B13}
\]
From equation (B13), we obtain
\[
P_n + Q_n = \begin{cases} -\frac{K_0 H_A A_L}{\pi}, & n = 1 \\ -\frac{K_0 H_A A_L}{(m\pi)}, & n = m \\ 0, & \text{others} \end{cases} \tag{B14}
\]
Applying equation (10b) to equation (B11),
\[
\psi(x, D) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi x}{L} \right) \left\{ P_n \exp \left[ \left( \frac{A}{2} + a_n \right) D \right] \right. \\
\left. + Q_n \exp \left[ \left( \frac{A}{2} - a_n \right) D \right] \right\}. \tag{B15}
\]
Therefore,
\[
P_n \exp \left[ \left( \frac{A}{2} + a_n \right) D \right] + Q_n \exp \left[ \left( \frac{A}{2} - a_n \right) D \right] = 0. \tag{B16}
\]
By solving equations (B14) and (B16), we get the values of $P_n$ and $Q_m$.
\[
P_n = \begin{cases} \frac{K_0 H_A A_L \exp(-a_1 D)}{\pi 2 \sinh(a_1 D)}, & n = 1 \\ \frac{K_0 H_A A_L \exp(-a_m D)}{m\pi 2 \sinh(a_m D)}, & n = m \\ 0, & \text{others} \end{cases} \tag{B17}
\]
\[
Q_n = \begin{cases} \frac{K_0 H_A A_L \exp(a_1 D)}{\pi 2 \sinh(a_1 D)}, & n = 1 \\ \frac{K_0 H_A A_L \exp(a_m D)}{m\pi 2 \sinh(a_m D)}, & n = m \\ 0, & \text{others} \end{cases} \tag{B18}
\]
[51] Applying equations (B17) and (B18) to equation (B11), we obtain the distribution of stream function:
\[
\psi(x, z) = \frac{K_0 L}{\pi} H_A A_L \sin \left( \frac{n\pi x}{L} \right) \exp \left( \frac{A}{2z} \right) \sinh \left[ a_1 \left( z - D \right) \right] \sinh \left( a_1 D \right) \\
+ \frac{K_0 L}{m\pi} H_A A_L \sin \left( \frac{m\pi x}{L} \right) \exp \left( \frac{A}{2z} \right) \sinh \left[ a_m \left( z - D \right) \right] \sinh \left( a_m D \right). \tag{B19}
\]

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