The sum numbers and
the integral sum numbers of $F_n$ and $F_n$

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Abstract  Let $G = (V, E)$ be a simple graph with the vertex set $V$
and the edge set $E$. $G$ is a sum graph if there exists a labelling $f$ of
the vertices of $G$ into distinct positive integers such that $uv \in E$
if and only if $f(w) = f(u) + f(v)$ for some vertex $w \in V$. Such a labelling $f$
is called a sum labelling of $G$. The sum number $\sigma(G)$ of $G$ is the smallest
number of isolated vertices which result in a sum graph when added to $G$.
Similarly, the integral sum graph and the integral sum number $\zeta(G)$ are
also defined. The difference is that the labels may be any distinct integers.
In this paper, we will determine that

$$
\left\{ \begin{array}{l}
0 = \zeta(F_5) < \sigma(F_5) = 1; \\
1 = \zeta(F_6) < \sigma(F_6) = 2; \\
3 = \zeta(F_7) < \sigma(F_7) = 4; \\
\zeta(F_n) = \sigma(F_n) = 0, \ n = 1, 2, 3; \\
\zeta(F_n) = \sigma(F_n) = 2n - 7, \ n \geq 7.
\end{array} \right.
$$

and

$$
\left\{ \begin{array}{l}
0 = \zeta(F_5) < \sigma(F_6) = 1; \\
2 = \zeta(F_6) < \sigma(F_6) = 3; \\
\zeta(F_n) = \sigma(F_n) = 0, \ n = 3, 4; \\
\zeta(F_n) = \sigma(F_n) = 2n - 8, \ n \geq 7.
\end{array} \right.
$$

Keywords  The sum graph; The integral sum graph; The sum
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1. Introduction

Let $G = (V, E)$ be a simple graph with the vertex set $V$ and the edge set $E$. The complement $\overline{G}$ of $G$ with order $n$ is the graph with the vertex set $V$ and the edge set $E(K_n) - E$. A path $P_n$ is a graph with the vertex set $\{a_1, a_2, ..., a_n\}$ and the edge set $\{a_1a_2, a_2a_3, ..., a_{n-1}a_n\}$, and $a_1$ and $a_n$ are called the end vertices of $P_n$. A fan $F_n$ is a graph with the vertex set $\{c, a_1, a_2, ..., a_n\}$ and the edge set $\{ca_1, ca_2, ..., ca_n\} \cup \{a_1a_2, a_2a_3, ..., a_{n-1}a_n\}$. It is obvious that $\overline{F_n} = \overline{P_n} \cup K_1$.

A sum graph and an integral sum graph were introduced by Frank Harary in [2] and [3]. $G$ is a sum graph if there exists a labelling $f$ of the vertices of $G$ into distinct positive integers such that $uv \in E$ if and only if $f(w) = f(u) + f(v)$ for some vertex $w \in V$. Such a labelling $f$ is called a sum labelling of $G$. A sum graph cannot be connected. There must always be at least one isolated vertex. The sum number $\sigma(G)$ of $G$ is the smallest number of isolated vertices which result in a sum graph when added to $G$. Similarly, an integral sum graph and an integral sum number $\zeta(G)$ are also defined. The difference is that the labels may be any distinct integers. Obviously $\zeta(G) \leq \sigma(G)$.

A vertex $w$ of $G$ is working if its label corresponds to an edge $uv$ of $G$. $G$ is exclusive if none of the vertices in $V$ is working. For example, $K_n$ and $W_{2n-1}$ are exclusive in [9].

To simplify the notations, we may assume that the vertices of $G$ are identified with their labels throughout this paper. And let $V_i$ and $E_i$ denote the set of the vertices independent of $a_i$ and the set of the edges adjacent to $a_i$ in $\overline{P_n}$ respectively. Besides, some results have been obtained as follow.

**Lemma 1** ([2]) $\sigma(P_n) = 1$ and $\zeta(P_n) = 0$ for $n \geq 2$.

**Lemma 2** ([1][8]) $\zeta(K_n) = \sigma(K_n) = 2n - 3$ for $n \geq 4$.

**Lemma 3** ([8]) $\zeta(C_n) = \zeta(W_n) = 0$ for $n \neq 5$.

**Lemma 4** ([2]) For $n \geq 3$, $\sigma(C_n) = \begin{cases} 2, & n \neq 4, \\ 3, & n = 4. \end{cases}$

**Lemma 5** ([10][7]) For $n \geq 3$, $\sigma(W_n) = \frac{n}{2} + 2$, $n$ even, 

$n$, $n$ odd.

In this paper, we will determine the sum numbers and the integral sum numbers of $\overline{P_n}$ and $\overline{F_n}$ for $n \geq 1$.

2. Main results

Let $\overline{P_n} = (V, E)$ and $S = V \cup C$, where $V = \{a_1, a_2, ..., a_n\}$ and $C$ is the isolated vertex set. It is clear that $\overline{P_2} = 2K_1$ and $\overline{P_3} = P_2 \cup K_1$ and $\overline{P_4} = P_3$. By Lemma 1 and Lemma 2, $\zeta(\overline{P_i}) = \sigma(\overline{P_i}) = 0$ for $i = 1, 2, 3$ and $0 = \zeta(\overline{P_i}) < \sigma(\overline{P_i}) = 1$. In this section, we only consider $n \geq 5$. 

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Lemma 2.1 \( \overline{P_n} \) is not an integral sum graph for \( n \geq 5 \).

**Proof:** Let \( |a_x| = \max\{|a| : a \in V\} \) and \( a_x \in V \). Assume that \( a_x > 0 \) (A similar argument work for \( a_x < 0 \)). By contradiction. If \( \overline{P_n} \) is an integral sum graph for \( n \geq 5 \) then \( 0 \not\in V \) and \( a_x + a_i \in V \) for all \( a_x a_i \in E \). Then \( a_x + a_i > 0 \) and \( a_i < 0 \) according to the choice of \( a_x \). So we get at least \( n - 3 \) distinct positive vertices \( a_x + a_i \) in \( V \). Meanwhile, we also get at least \( n - 3 \) distinct negative vertices \( a_i \). So \( 2(n - 3) + 1 \leq n \), that is, \( n \leq 5 \). Since \( n \geq 5 \), only \( n = 5 \).

Assume that \( V(\overline{P_5}) = \{a_x, a_1, a_2, a_x + a_1, a_x + a_2\} \) with \( a_i < 0 \) \( (i = 1, 2) \). By the choice of \( a_x \), \( (a_x + a_i)a_x \not\in E \) with \( i = 1, 2 \) and \( \overline{V_x} = \{a_x + a_1, a_x + a_2\} \) (see Figure 1). So \( (a_x + a_1)(a_x + a_2) \in E \), that is, \( (a_x + a_1) + (a_x + a_2) \in \{a_x, a_x + a_1, a_x + a_2\} \). Thus, \( (a_x + a_1) + (a_x + a_2) = a_x \), that is, \( a_x + a_1 = -a_2 \) and \( a_x + a_2 = -a_1 \). So \( (a_x + a_1)a_2 \not\in E \) and \( (a_x + a_2)a_1 \not\in E \) and \( a_1 a_2 \not\in E \) (see Figure 1), contracting the structure of \( \overline{P_5} \).

Thus, Lemma 2.1 holds. \( \square \)

![Figure 1](image1.png)

Lemma 2.2 \( \zeta(\overline{P_5}) = 1 \).

**Proof:** By Lemma 2.1, \( \zeta(\overline{P_5}) \geq 1 \). Below we will give an integral sum labelling of \( \overline{P_5} \cup K_1 \) (see Figure 2). So \( \zeta(\overline{P_5}) \leq 1 \). Thus, \( \zeta(\overline{P_5}) = 1 \). \( \square \)

![Figure 2](image2.png)

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Lemma 2.3 \( \sigma(P_5) = 2 \).

**Proof:** It is clear that the sum number of a graph must be at least as large as the minimum degree of the graph, so \( \sigma(P_5) \geq 2 \). Figure 3 below shows that \( \sigma(P_5) \leq 2 \). Thus, \( \sigma(P_5) = 2 \). \( \square \)

![Figure 3](image)

Lemma 2.4 If \( a_x \in V \) with \( |a_x| = \max\{|a| : a \in V\} \), then there exists one edge \( a_xa_j \in E \) such that \( a_x + a_j \in C \) for \( n \geq 6 \).

**Proof:** Let \( |a_x| = \max\{|a| : a \in V\} \) with \( a_x \in V \). Assume that \( a_x > 0 \) (A similar argument works for \( a_x < 0 \)). By contradiction. Suppose to the contrary that \( a_x + a_j \in V \) for all \( a_xa_j \in E \). According to the choice of \( a_x \), \( a_x + a_j > 0 \) and \( a_j < 0 \). Then there are at least \( n - 3 \) distinct positive vertices \( a_x + a_j \) and \( n - 3 \) distinct negative vertices adjacent to \( a_x \). So \( (n - 3) + (n - 3) + 1 \leq n \), i.e., \( n \leq 5 \), contradicting the fact \( n \geq 6 \).

Thus, Lemma 2.4 holds. \( \square \)

Lemma 2.5 \( \zeta(P_6) = 3 \).

**Proof:** Below we give an integral sum labelling of \( P_6 \) (see Figure 4). So \( \zeta(P_6) \leq 3 \). What we need to do is only to prove \( \zeta(P_6) \geq 3 \).

![Figure 4](image)
Let $|a_x| = \max\{|a| : a \in V\}$ with $a_x \in V$. Assume that $a_x > 0$ (A similar argument works for $a_x < 0$). By Lemma 2.2, there exists one edge $a_x a_{j_0} \in E_x$ such that $a_x + a_{j_0} \in C$. Firstly, we will show Claim 1 and Claim 2 and Claim 3.

**Claim 1:** There exists another edge $a_x a_l \in E_x - a_x a_{j_0}$ such that $a_x + a_l \in C$.

By contradiction. Suppose to the contrary that $a_x + a_l \in V$ for all $a_x a_l \in E_x - a_x a_{j_0}$. By the choice of $a_x$, $a_x + a_l > 0$ and $a_l < 0$. Since $a_x + a_{j_0} \in C$, $(a_x + a_{j_0}) + a_l = (a_x + a_l) + a_{j_0} \notin S$. Then $a_x + a_l \in \{a_{j_0}\} \cup V_{j_0}$, denoted (1).

If $a_x$ is an end vertex of $P_6$, then $|E_x| = 4$. So there are four distinct positive vertices and at least three distinct negative vertices in $V$. So $n > 6$, a contraction. So $a_x$ is not an end vertex of $P_6$.

Let $V_x = \{a_{x+1}, a_{x-1}\}$ and $V = \{a_x, a_{x+1}, a_{x-1}, a_{j_0}, a_{i_1}, a_{i_2}\}$ (see Figure 5). Assume $a_i, a_j$ are two end vertices of $P_6$. Then $a_i a_j \in E$ and $a_i + a_j \in S$.

![Figure 5](image)

If $V_{j_0} \subseteq \{a_{i_1}, a_{i_2}\}$ and $a_i a_j \in \{a_{x+1} a_{i_1}, a_{j_0} a_{i_2}, a_{i_2} a_{x-1}\}$ then we may assume that $a_i a_{j_0} \notin E$ (see Figure 6). By (1), $a_x + a_{i_1} = a_{j_0}$ and $a_x + a_{i_2} = a_{j_0}$, a contraction. So $V_{j_0} \subseteq \{a_{i_1}, a_{x+1}\}$.

![Figure 6](image)
By (1) and $a_{l_1} < 0$, $\{a_x + a_{l_1}, a_x + a_{l_2}\} = \{a_{j_0}, a_{x+1}\}$ and $a_{x+1}a_{j_0} \notin E$ (see Figure 7).

If $a_x + a_{l_1} = a_{x+1}$ and $a_x + a_{l_2} = a_{j_0}$ then $a_{l_1} + a_{j_0} = a_{x+1} + a_{l_2} \in S$. So $a_{l_1}a_{j_0} \in E$. Since $a_{x-1} + a_{j_0} = a_{x-1} + (a_x + a_{l_2}) = a_x + (a_{x-1} + a_{l_2}) \in S$, $a_{x-1} + a_{l_2} = a_{l_1}$, contracting $a_{x-1}a_{l_2} \notin E$.

If $a_x + a_{l_1} = a_{j_0}$ and $a_x + a_{l_2} = a_{x+1}$ then $a_{l_1} + a_{x+1} = a_{j_0} + a_{l_2} \in S$. Uniting $a_{l_1} + a_{x+1} = a_{j_0} + a_{l_2} \in S$ and $a_{x-1} + a_{l_1} = a_{l_2}$, we have $a_{j_0} + a_{x-1} = a_{x+1}$. If $a_{l_1} + a_{x+1} = a_{j_0} + a_{l_2} \in V$ then $a_{l_1} + a_{x+1} = a_{j_0} + a_{l_2} = a_{x-1}$. Since $a_{x-1} = a_{j_0} + a_{l_2} = a_{j_0} + (a_{x-1} + a_{l_1})$, $a_{j_0} + a_{l_1} = 0$. Thus, $(a_x + a_{j_0}) + a_{l_1} = a_x + (a_{j_0} + a_{l_1}) = a_x \in S$, contracting $a_x + a_{j_0} \in C$. If $a_{l_1} + a_{x+1} = a_{j_0} + a_{l_2} \in C$ then $a_{x+1} + a_{l_2} = (a_{j_0} + a_{x-1}) + a_{l_2} = (a_{j_0} + a_{l_2}) + a_{x-1} \in S$, contracting $a_{j_0} + a_{l_2} \in C$.

Thus, Claim 1 holds.

Up to now, we may assume that $a_x + a_{j_0} \in C$ and $a_x + a_{l_1} \in C$ with $\{a_xa_{j_0}, a_xa_{l_1}\} \subseteq E_x$.

**Claim 2:** If $a_x$ is one end vertex of $P_6$ then $\zeta(P_6) \geq 3$.

In fact, if $a_x$ is an end vertex of $P_6$ then $|V_x| = 1$. Let $V_x = \{a_{x+1}\}$. Assume that $a_{j_0}a_{x-1} \in E$ (see Figure 8) (if not, we can consider $a_{l_1}a_{x-1} \in E$).
By contradiction. Suppose to the contrary that $\zeta(P_6) \leq 2$. By Claim 1, $\zeta(P_6) \geq 2$. So $\zeta(P_6) = 2$. Let $C = \{a_x + a_j, a_x + a_i\}$. Then $\{a_x + a_{x-1}, a_x + a_{i_2}\} \subseteq \{(a_{j_0}) \cup V_x) \cap (\{a_{l_1}\} \cup V_{l_1})$.

According to the choice of $a_{x-1} < 0$ and $a_{i_2} < 0$. So there is only one case of $\{a_x + a_{x-1}, a_x + a_{i_2}\} = \{a_{j_0}, a_{l_1}\}$ and $P_6 = a_{x}a_{x+1}a_{j_0}a_{i_2}a_{1}a_{x-1}$ (see Figure 9).

![Figure 9](image)

If $a_x + a_{x-1} = a_{j_0}$ and $a_x + a_{i_2} = a_{l_1}$ then $a_{j_0} + a_{l_1} = (a_x + a_{x-1}) + a_{l_1} = (a_x + a_{l_1}) + a_{x-1} \in S$, contracting $a_x + a_{l_1} \in C$.

If $a_x + a_{x-1} = a_{l_1}$ and $a_x + a_{i_2} = a_{j_0}$ then $a_{j_0} + a_{l_1} = (a_x + a_{i_2}) + a_{j_0} = (a_x + a_{j_0}) + a_{l_1} \in S$, contracting $a_x + a_{j_0} \in C$.

Thus, Claim 2 holds.

**Claim 3:** If $a_x$ is not an end vertex of $P_6$ then $\zeta(P_6) \geq 3$.

In fact, if $a_x$ is not an end vertex of $P_6$ then $|V_x| = 2$. Let $V_x = \{a_{x+1}, a_{x-1}\}$. Then $a_xa_{x+1} \notin E$ and $a_xa_{x-1} \notin E$. Let $a_xa_{i_2} \in E_x - \{a_x, a_{j_0}, a_xa_{l_1}\}$. Since $\{a_x + a_{j_0}, a_x + a_{l_1}\} \subseteq C$, $a_x + a_{i_2} \in (\{a_{j_0}\} \cup V_{j_0}) \cap (\{a_{l_1}\} \cup V_{l_1}) \cup C$.

By contradiction. Suppose to the contrary that $\zeta(P_6) \leq 2$. By Claim 1, $\zeta(P_6) = 2$. Let $C = \{a_x + a_{j_0}, a_x + a_{l_1}\}$. Then $a_x + a_{i_2} \in V$. So $a_{i_2} < 0$ and it is impossible that both of $a_{j_0}$ and $a_{l_1}$ are adjacent to $a_{i_2}$. Assume $a_ja_i \in \{a_{x+1}, a_{j_0}, a_{j_0}a_{i_1}, a_{i_1}a_{l_2}, a_{l_2}a_{x-1}\}$. Then $a_x + a_{i_2} \in \{a_{j_0}, a_{l_1}\}$ with $a_{j_0}a_{i_1} \notin E$ (if $a_{j_0}a_{i_1} \in E$ then $a_x + a_{i_2} \in (\{a_{j_0}\} \cup V_{j_0}) \cap (\{a_{l_1}\} \cup V_{l_1}) = \emptyset$. It is impossible.) (see Figure 10).
Similarly, \( a_{j_0} + a_{x-1} \in \{a_x, a_{x+1}\} \cup C \); \( a_{x+1} + a_{l_1} \in \{a_x, a_{x-1}\} \cup C \); \( a_{x-1} + a_{l_1} \in \{a_x, a_{x+1}\} \cup C \); \( a_{x+1} + a_{l_2} \in \{a_x, a_{x-1}\} \cup C \); \( a_{j_0} + a_{l_2} \in \{a_x, a_{x+1}\} \cup C \). (1.1) If \( a_{x} + a_{l_2} = a_{j_0} \) and \( a_{j_0} + a_{x-1} = a_{x} \) then \( a_{x} + a_{l_1} = (a_{j_0} + a_{x-1}) + a_{l_1} = a_{j_0} + (a_{x-1} + a_{l_1}) \in S \). So \( a_{x-1} + a_{l_1} = a_{x+1} \), which implies \( a_{x} + a_{l_1} = a_{j_0} + a_{x-1} \) and \( a_{j_0} a_{x+1} + E \). Since \( (a_{x} + a_{j_0}) + a_{x+1} = a_{x} + (a_{j_0} + a_{x+1}) \not\in S \), \( a_{j_0} + a_{x+1} \in \{a_{x-1}\} \cup C \).

(1.1.1) If \( a_{j_0} + a_{x+1} = a_{x-1} \) then \( a_{l_1} + a_{j_0} = 0 \) (since \( a_{x-1} + a_{l_1} = a_{x+1} \)). So \( a_{x} + a_{l_1} = (a_{j_0} + a_{x-1}) - a_{j_0} = a_{x-1} \in V \), contracting \( a_{x} + a_{l_1} \in C \).

(1.1.2) If \( a_{x+1} + a_{j_0} \in C \), \( (a_{x+1} + a_{j_0}) + a_{l_1} = (a_{x+1} + a_{l_1}) + a_{j_0} \not\in S \). Then \( a_{x+1} + a_{l_1} \in C \). So \( \{a_{x+1} + a_{j_0}, a_{x+1} + a_{l_1}\} \subseteq C = \{a_{x} + a_{j_0}, a_{x} + a_{l_1}\} \), a contraction.

(1.2) If \( a_{x} + a_{l_2} = a_{j_0} \) and \( a_{j_0} + a_{x-1} = a_{x+1} \) then \( a_{x+1} + a_{l_1} = (a_{j_0} + a_{x-1} + a_{l_1} = a_{j_0} + a_{x+1}) \in S \). So \( a_{x-1} + a_{l_1} = a_{x} \). Since \( a_{x+1} + a_{l_2} = a_{j_0} + a_{x-1} = a_{x} + a_{l_1} + a_{l_2} = a_{j_0} + a_{l_2} = a_{x+1} \). Uniting \( a_{x} + a_{l_2} = a_{j_0} \) and \( a_{j_0} + a_{l_2} = a_{x+1} \), we have \( a_{x} + 2a_{l_2} = a_{x-1} \), contracting the choice of \( a_{x} \).

(1.3) If \( a_{x} + a_{l_2} = a_{j_0} \) and \( a_{j_0} + a_{x-1} \in C \) then \( a_{l_2} = a_{j_0} + a_{x-1} \) \( a_{j_0} + a_{x-1} \not\in S \). So \( a_{l_2} + a_{j_0} = a_{x-1} \). If not, then \( a_{l_2} + a_{j_0} \in C \), but \( a_{l_2} + a_{j_0} \not\in C = \{a_{x} + a_{j_0}, a_{x} + a_{l_1}\} \), a contraction.). Then \( (a_{x+1} + a_{l_2}) + a_{j_0} = a_{x+1} + a_{x-1} \in S \). So \( a_{x+1} + a_{l_2} = a_{x} \), contracting the choice of \( a_{x} \).

(2) If \( a_{x} + a_{l_2} = a_{j_0} \) then \( a_{l_2} + a_{x+1} = a_{x} + a_{l_2} + a_{x+1} = a_{x} + (a_{x} + a_{l_2}) + a_{x+1} \in S \). So \( a_{x+1} + a_{l_2} = a_{x} + a_{l_2} \), \( a_{x} + a_{l_2} = a_{x}, \) contracting \( a_{x} + a_{l_2} \not\in C \).

Assume that \( a_{x-1} = x \). By the above, \( a_{j_0} = 2x \) and \( a_{x+1} = 3x \) and \( a_{x} = 5x \).

(2.1) If \( a_{x} + a_{l_2} = a_{x-1} \) (note that \( a_{x} + a_{x-1} = a_{x+1} \)). Assume that \( a_{x-1} = x \). By the above, \( a_{j_0} = 2x \) and \( a_{x+1} = 3x \), contracting \( a_{x} + a_{j_0} \not\in E \).

(2.2) If \( a_{x} + a_{x-1} = 2a_{l_2} = a_{j_0} + a_{x-1} = a_{x+1} \) (If \( a_{j_0} + a_{x-1} \in C \) then \( (a_{j_0} + a_{x-1}) + a_{x+1} = a_{j_0} + (a_{x-1} + a_{x+1}) = a_{j_0} + a_{l_2} \) \( a_{j_0} + a_{l_2} \), a contraction). Since \( a_{j_0} + a_{x+1} = a_{j_0} + a_{x-1}, a_{j_0} + a_{l_2} = a_{j_0} + a_{l_2} \in C \).

(2.2.1) If \( a_{j_0} + a_{l_2} = a_{x-1} \) then \( -a_{l_2} = a_{x-1} - a_{j_0} + a_{l_2} = a_{l_2} + a_{x+1} + a_{l_2} \) contracting \( a_{x+1} = 2a_{l_2} \).

(2.2.2) If \( a_{j_0} + a_{l_2} = a_{x-1} \) then \( a_{j_0} + a_{l_2} = a_{x+1} \) \( a_{j_0} + a_{l_2} = a_{x} + a_{l_2} = 2a_{x} + a_{l_2} \). So \( a_{j_0} = 2a_{x} \), contracting the choice of \( a_{x} \).

Therefore, Claim 3 holds.

Thus, Lemma 2.5 holds. \( \Box \)
Lemma 2.6 \( \sigma(P_6) = 4 \).

Proof: Let \( V = \{a_x, a_{x+1}, a_{x-1}, a_{l_1}, a_j, a_{j_0}\} \) and \( a_x = \max \{a : a \in V\} \). Then \( a_x + a_i \in C \) for all \( a_x, a_i \in E \). Firstly, Figure 11 below shows that \( \sigma(P_6) \leq 4 \). What we need is to prove \( \sigma(P_6) \geq 4 \).

![Figure 11](image)

If \( a_x \) is an end vertex of \( P_n \) then \( \sigma(P_6) \geq 4 \). Otherwise, it is clear that \( \sigma(P_6) \geq 3 \). Below we will prove that \( \sigma(P_6) \neq 3 \).

By contradiction. Suppose to the contrary that \( \sigma(P_6) = 3 \). Then \( C = \{a_x + a_{l_1}, a_x + a_{l_2}, a_x + a_{j_0}\} \). Assume that \( V_x = \{a_{x+1}, a_{x-1}\} \) and \( a_i, a_j \in \{a_{l_1}, a_{x+1}, a_{x-1}, a_{j_0}\} \), where \( a_i \) and \( a_j \) are two end vertices of \( P_6 \) (see Figure 12).

![Figure 12](image)

Since \( (a_x + a_{j_0}) + a_{l_1} = a_x + (a_{j_0} + a_{l_1}) \not\in S \), \( a_{j_0} + a_{l_1} \in \{a_x, a_{x+1}, a_{x-1}\} \cup C \). Similarly, \( a_{x-1} + a_{l_1} \in \{a_x, a_{x+1}\} \cup C \); \( a_{x-1} + a_{l_2} \in \{a_x, a_{x+1}\} \cup C \); \( a_{x+1} + a_{l_2} \in \{a_x, a_{x-1}\} \cup C \); \( a_{x+1} + a_{j_0} \in \{a_x, a_{x-1}\} \cup C \).

(I) If \( a_{j_0} + a_{l_1} = a_x \) then \( a_x + a_{x-1} = (a_{j_0} + a_{l_1}) + a_{x-1} = (a_{x-1} + a_{l_1}) + a_{j_0} \not\in S \), which implies \( a_{x-1} + a_{l_1} \in \{a_{j_0}\} \cup \{a_{l_1}, a_{l_2}\} \cup C \). By the above, \( a_{x-1} + a_{l_1} \in C \). So
\((a_{x-1} + a_{i_1}) + a_{i_2} = a_{i_1} + (a_{x-1} + a_{i_2}) \not\in S\), which implies \(a_{x-1} + a_{i_2} \not\in \{a_{x+1}\} \cup C\).

Similarly, \(a_{x-1} + a_{i_2} \in \{a_{i_1}\} \cup \overline{V_i} \cup C\) with \(\overline{V_i} \subseteq \{a_{x+1}, a_{i_2}\}\).

(1.1) If \(a_{x-1} + a_{i_2} = a_{x+1}\) then \(a_{x-1} + a_{i_2} \in \{a_{i_1}\} \cup C\). Furthermore, \(a_{x-1} + a_{i_2} \in C\). (If not, then \(a_{x-1} + a_{x+1} = a_{i_1}\). Then \(a_x = a_{x_0} + a_{i_1} = a_{x_0} + (a_{x-1} + a_{x+1}) = (a_{x_0} + a_{i_1}) + a_{x-1} \not\in S\). So \(a_{x_0} + a_{i_1} = a_{i_1} = a_{x-1} + a_{x+1}\), which implies \(a_{x_0} + a_{i_2} = a_{x-1} + a_{x+1}\), a contradiction with \(a_{x-1} + a_{i_2} = a_{x+1}\).)

So \((a_{x-1} + a_{i_2}) + a_{j_1} = a_{x-1} + (a_{i_2} + a_{j_1}) \not\in S\). By the above, \(a_{x+1} + a_{j_1} \in \{a_{x}\} \cup C\).

(1.1.1) If \(a_{x+1} + a_{j_1} = a_{x}\) then \(a_{x} + a_{j_0} = (a_{x+1} + a_{i_2}) + a_{j_0} = (a_{x+1} + a_{i_2}) + a_{j_0} \not\in S\). which implies \(a_{x+1} + a_{j_0} \not\in V\). By the above, \(a_{x+1} + a_{j_0} \not\in a_{x-1}\).

Uniting \(a_{x+1} = a_{x-1} + a_{i_2}\), we have \(a_{i_2} + a_{j_0} = 0\), a contradiction.

(1.1.2) If \(a_{x+1} + a_{j_1} \in C\) then \((a_{x+1} + a_{j_1}) + a_{j_0} = (a_{x+1} + a_{j_0}) + a_{j_1} \not\in S\), which implies \(a_{x+1} + a_{j_0} \not\in \{a_{i_1}\} \cup \overline{V_i} \cup C\). By the above, \(a_{x+1} + a_{j_0} \in C\).

(1.1.2.1) If \(a_{x+1} + a_{i_1} \in E\) then \(a_{x+1} + a_{i_1} \in \{a_{i_1}\} \cup \overline{V_i} \cup C\). By the above, \(a_{x+1} + a_{i_1} \in C\).

(1.1.2.2) If \(a_{i_1} + a_{i_2} \in E\) then \(a_{i_1} + a_{i_2} \in \{a_{i_1}\} \cup \overline{V_i} \cup C\). Uniting \(a_{i_1} + a_{i_2} \in \{a_{i_1}\} \cup \overline{V_i} \cup C\), we have \(a_{i_1} + a_{i_2} = a_{i_1}\). So \(a_{x+1} + a_{i_2} = a_{x} + a_{i_1} = a_{x} + a_{i_1} \in C\), contradicting \(a_{x+1} + a_{i_2} \in C\).

(1.1.2.3) If \(a_{i_1} + a_{i_1} \in E\) then \(a_{i_1} + a_{i_1} \in S\). Since \(a_{x+1} + a_{i_1} = (a_{x_0} + a_{i_1}) + a_{i_2} = (a_{x_0} + a_{i_1}) \not\in S\), we have \(a_{x+1} + a_{i_1} = a_{x_0} + a_{i_1} \not\in S\). So \(a_{x+1} + a_{i_1} = (a_{x+1} + a_{i_1} + a_{i_1}) \not\in S\), contradicting \(a_{x+1} + a_{i_1} \in C\).

(1.1.2.4) If \(a_{x-1} + a_{j_0} \in E\) then \(a_{x-1} + a_{j_0} = (a_{x-1} + a_{i_2}) + a_{j_0} = (a_{x-1} + a_{j_0}) + a_{i_2} \not\in S\). So \(a_{x-1} + a_{j_0} = (a_{x-1} + a_{j_0}) + a_{i_2} \not\in S\), contradicting \(a_{x-1} + a_{j_0} \in S\).

(1.2) If \(a_{x+1} + a_{i_2} \in C\) then \(a_{x+1} + a_{i_2} \in \{a_{i_1}\} \cup \overline{V_i} \cup C\). Since \(a_{x-1} + a_{i_1} \in C\), \(a_{x-1} + a_{x-1} \in \{a_{i_1}\} \cup \overline{V_i} \cup C\). So \(a_{x-1} + a_{x-1} \in \{a_{i_1}\} \cup \overline{V_i} \cup C\).

Since \(a_{x+1} + a_{j_0} \not\in a_{x} + a_{i_1}, a_{x+1} + a_{j_0} \in \{a_{x-1}\} \cup C\). Since \(a_{x+1} + a_{j_0} \not\in a_{x} + a_{i_1}, a_{x-1} \not\in \{a_{x-1}\} \cup C\).

(1.2.1) If \(a_{x-1} + a_{i_1} = a_{x} + a_{i_2} \) then \(a_{x-1} = a_{j_0} + a_{i_2}\) and \(a_{x} + a_{i_2} = a_{i_1} + a_{j_0}\). So \(2a_{i_2} = a_{j_0} + a_{i_2} = a_{x}\). Since \(a_{x+1} + a_{j_0} \not\in a_{x} = a_{i_2} + a_{x+1} + a_{j_0} \in C\). Uniting \(a_{x} + a_{i_2} = a_{x} + a_{i_1}\), we have \(a_{x+1} + a_{i_1} = a_{i_1}\) and \(a_{x} + a_{i_2} = a_{x} + a_{i_1}\).

(1.2.2) If \(a_{x-1} + a_{j_0} = a_{x} + a_{i_1}\) then \(a_{x-1} + a_{j_0} = a_{x} + a_{i_1}\) and \(a_{x} + a_{j_0} = a_{x} + a_{i_1}\).

(1.2.2.1) If \(a_{x+1} + a_{j_0} = a_{x} + a_{i_1}\) then \(a_{x+1} = 2a_{i_1}\). So \(2a_{i_1} = a_{j_0} + a_{i_1} \not\in a_{x} + a_{i_1}\). By the choice of \(a_{x}, a_{x+1} + a_{i_1} \in C\). Since \(a_{x} + a_{x+1} = 2a_{i_1} + 2a_{j_0} = 2(a_{j_0} + a_{i_1}) = 2a_{i_1} \in S\), contradicting the choice of \(a_{x}\).

(1.2.2.2) If \(a_{x+1} + a_{j_0} = a_{x} + a_{i_1}\) then \(a_{x+1} = 2a_{i_1}\) and \(a_{x} + a_{j_0} = 2a_{i_1} + 2a_{j_0} = 2(a_{j_0} + a_{i_1}) = 2a_{i_1} \not\in a_{x} + a_{i_1}\), a contradiction.
By Lemma 2.4, there exists one edge $=2a_0 + a_1 = a_{x+1}$ then $a_{x+1} = (a_{j_0} + a_{l_0}) + a_{x-1} = (a_{j_0} + a_{l_0}) + a_{x-1} + a_{l_1} \in S$. Then $a_{x-1} + a_{j_0} = a_x$. Since $a_{x+1} + a_{l_1} = (a_{x-1} + a_{j_0}) + a_{l_2} = (a_{x-1} + a_{l_2} + a_{j_0} \in S, a_{x-1} + a_{l_2} = a_{x+1} = (a_{j_0} + a_{l_1})$.

Uniting $a_{x-1} + a_{j_0} = a_x$ and $a_{x-1} + a_{l_2} = a_{x+1}$, we have $a_x + a_{l_2} = a_{x+1} + a_{j_0} \in C$.

Uniting $a_{x-1} + a_{j_0} = a_x$ and $a_{j_0} + a_{l_1} = a_{x+1}$, we have $a_x + a_{l_1} = a_{x+1} + a_{x-1} \in C$.

Uniting $(a_{x+1} + a_{j_0}) + a_{l_2} = (a_{x+1} + a_{l_2}) + a_{j_0} \notin S$ and $a_{x-1} + a_{l_2} = a_{x+1}$, we have $a_{x+1} + a_{l_2} \in C$. Then $a_{x+1} + a_{l_2} = a_x + a_{j_0}$, that is, $(a_{x-1} + a_{j_0}) + a_{j_0} = (a_{x-1} + a_{j_0}) + a_{l_2}$. So $2a_{j_0} = 2a_{l_2}$, a contradiction.

A similar argument works for $a_{j_0} + a_{l_1} = a_{x+1}$.

(III) If $a_{j_0} + a_{l_1} \in C$ then $a_{j_0} + a_{l_1} = a_x + a_{l_2}$.

Since $(a_{j_0} + a_{l_1} + a_{x-1} = (a_{j_0} + a_{l_1} + a_{j_0} \notin S, a_{l_1} + a_{x-1} \in C$. So $a_{l_1} + a_{x-1} = a_x + a_{l_2}$.

Since $a_{j_0} + a_{l_1} \in C, (a_{j_0} + a_{l_1} + a_{x-1} = (a_{j_0} + a_{l_1} + a_{j_0} \notin S, a_{j_0} + a_{x+1} = a_x + a_{j_0} \in C$.

Since $a_{j_0} + a_{x-1} \in C, (a_{j_0} + a_{x-1} + a_{x-1} = a_{j_0} + (a_{x-1} + a_{x-1}) \notin S$. So $a_{x-1} + a_{x-1} = a_{x} + a_{l_2} = a_{x-1} + a_{l_1} \in C$.

Uniting $a_{j_0} + a_{x-1} = a_x + a_{l_1}$ and $a_{x-1} + a_{x-1} = a_x + a_{l_2}$, we have $a_{x-1} + a_{l_1} = a_{j_0} + a_{l_2}$. Then $a_{j_0} + a_{l_2} \in E$.

Uniting $a_{x-1} + a_{x-1} = a_x + a_{l_2}$ and $a_{l_1} + a_{x-1} = a_x + a_{j_0}$, we have $a_{l_1} + a_{l_2} = a_{x+1} + a_{j_0} \in S$. Then $a_{l_1} + a_{l_2} \in E$, a contradiction.

So $\sigma(P_6) \neq 3$.

Thus, Lemma 2.6 holds. □

**Lemma 2.7** Let $|a_x| = \max\{|a| : a \in V\}$ with $a_x \in V$. Then $a_x + a_p \in C$ for all $a_x a_p \in E$ with $n = 7$.

**Proof:** Let $|a_x| = \max\{|a| : a \in V\}$ with $a_x \in V$. Assume that $V = \{a_x, a_{x+1}, a_{x-1}, a_{j_0}, a_{l_0}, a_{l_1}, a_{l_2}, a_{j_1}, a_{l_3}\}$ and $a_x > 0$ (A similar argument works for $a_x < 0$).

By Lemma 2.4, there exists one edge $a_x a_{j_0} \in E_x$ such that $a_x + a_{j_0} \in C$.

Since $(a_x + a_{j_0}) + a_{l_1} = (a_x + a_{l_1}) + a_{j_0} \notin S, a_x + a_{l_1} \in \{a_{j_0}, a_{l_1} \} \cup \{a_{j_2}, a_{l_2}\} \cup C$ for all $a_x a_{l_1} \in E_x - a_x a_{j_0}$.

**Claim** There exists at least one edge $a_x a_{l_1} \in E_x - a_x a_{j_0}$ such that $a_x a_{l_1} \in C$.

In fact, if $|V_x| = 2$ then $|E_x| = 5$. Since $|a_{j_0}, a_{j_1} \cup a_{j_0}| = 3$, Claim 1 holds. If $|V_x| = 1$ then suppose to the contrary that $a_x + a_{l_1} \in V$ for all $a_x a_{l_1} \in E_x - a_x a_{j_0}$, then $a_1 < 0$ and $a_x + a_{l_1} > 0$. So there are at least eight distinct vertices, contradicting $n = 7$.

Thus, Claim holds.

Assume that $a_x a_{x+1} \notin E$ and $a_x + a_{l_1} \in C$ with $a_x a_{l_1} \in E_x - a_x a_{j_0}$. Since $a_x a_{l_1} \in C, a_x + a_{l_1} \in \{a_{j_0}, a_{l_2} \} \cup \{a_{j_1}, a_{l_3}\} \cup C$ for all $a_x a_{l_1} \in E_x - \{a_x a_{j_0}, a_x a_{l_1}\}$ with $i = 2, 3$. So $a_x + a_{l_1} \in \{(a_{j_0}, a_{j_1} \cup a_{j_2}) \cap (a_{j_0}, a_{l_1} \cup a_{l_2}) \} \cup C$.

By contradiction. Suppose to the contrary that $a_x a_{l_1} \in V$. By the above, $a_x + a_{l_1} \in (a_{j_0} \cup a_{j_1}) \cap (a_{j_0} \cup a_{l_1})$. There are at most two cases (I) (II) in
all (see Figure 13, 15). Let \( a_i \) and \( a_j \) be two end vertices of \( P_7 \).

(I) If \( a_i + a_j \in \{a_{x+1}a_j, a_{j0}, a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5}, a_{i6}, a_{i7}, a_{i8} \} \) then \( a_x + a_l = 0 \in \{a_{j0}, a_{i1} \} \) (see Figure 13).

(I.1) If \( a_x + a_l = a_{j0} \) then \( a_{j0} + a_{i3} = (a_x + a_{i3}) + a_{i2} \in S \). So \( a_x + a_{i3} \in V \). Uniting \( a_x + a_{j0} \in C \) and \( a_x + a_{i3} \in C \), we have \( a_x + a_{i3} = a_{i1} \), which implies \( a_{i1} + a_{i2} = a_{i3} + a_{j0} \) and \( a_{i3} \leq 0 \). Then \( a_{i1} a_{i2} \in E \) (see Figure 14).

Since \( a_{j0} + a_{x-1} = (a_x + a_{i2}) + a_{x-1} = a_x + (a_{i2} + a_{x-1}) \in S \), \( a_{i2} + a_{x-1} \in \{a_{i1}, a_{i3} \} \).

Since \( a_{j0} + a_{x+1} = (a_x + a_{i2}) + a_{x+1} = a_x + (a_{i2} + a_{x+1}) \notin S \), \( a_{i2} + a_{x+1} \in \{a_{x-1} \} \cup C \).

If \( a_{i2} + a_{x+1} \in C \) then \( (a_{i2} + a_{x+1}) + a_{x-1} = (a_{i2} + a_{x-1}) + a_{x+1} \notin S \), contradicting \( a_{i2} + a_{x-1} \in \{a_{i1}, a_{i3} \} \). So \( a_{i2} + a_{x+1} = a_{x-1} \).

Since \( a_{i3} + a_{x-1} = a_{i3} + (a_{x+1} + a_{i0}) = a_{i2} + (a_{i3} + a_{x+1}) \notin S \) and \( a_{i3} + a_{x+1} \in \{a_{j0}, a_{i2} \}, a_{i3} + a_{x+1} = a_{i2} \). Note that \( a_{i1} + a_{i2} = a_{i3} + a_{j0} \in \{a_{x+1}, a_{x-1} \} \cup C \), then \( (a_{i1} + a_{i2}) = a_{i3} + a_{j0} \in C \). So \( (a_{i1} + a_{j0}) + a_{x+1} = (a_{i3} + a_{x+1}) + a_{j0} = a_{i2} + a_{j0} \in S \), contradicting \( a_{i3} + a_{j0} \in C \).

(I.2) If \( a_x + a_{i2} = a_{i1} \) then \( a_x + a_{i3} = a_{j0} \), which implies \( a_{i1} + a_{i3} = a_{i2} + a_{j0} \). So \( a_x + a_{i3} = a_{j0} \) (If not, then \( a_x + a_{i3} \in C \). So \( (a_x + a_{i3}) + a_{i2} = (a_x + a_{i2}) + a_{i3} = a_{i1} + a_{i3} \in S \), contradicting \( a_x + a_{i3} \in C \).)

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Since \( a_{l_1} + a_{x-1} = (a_x + a_{l_2}) + a_{x-1} = a_x + (a_{x-1} + a_{l_2}) \in S, a_{x-1} + a_{l_2} \in \{a_{j_0}, a_{l_1}\} \).

(I.2.1) If \( a_{x-1} + a_{l_2} = a_{j_0} \), then \( a_x + a_{j_0} = a_{l_1} + a_{x-1} \in C \). So \( a_{l_1} + a_{j_0} = a_{l_1} + (a_{x-1} + a_{l_2}) = (a_{l_1} + a_{x-1}) + a_{l_2} \in S \), contradicting \( a_{l_1} + a_{x-1} \in C \).

(I.2.2) If \( a_{x-1} + a_{l_2} = a_{l_1} \) then \( a_{x-1} + a_{l_1} = a_{j_0} \) (since \( a_{l_1} + a_{l_3} = a_{l_2} + a_{j_0} \)).

So \( a_x + a_{j_0} = a_x + (a_{x-1} + a_{l_1}) = (a_x + a_{l_1}) + a_{x-1} \in S \), contradicting \( a_x + a_{l_1} \in C \).

(II) If \( a_x a_j \in \{a_{x-1}a_{j_0}, a_{j_0}a_{l_1}, a_{l_1}a_{l_2}, a_{l_2}a_{x-1}, a_{x-1}a_x\} \) then \( a_x + a_{l_2} = a_{l_3} \) and \( a_{l_1}a_{l_3} \notin E \) and \( a_{j_0}a_{l_3} \notin E \). According to the choice of \( a_x, a_x + a_{l_3} \in C \) and \( a_{l_3} > 0 \) (see Figure 15).

II: Figure 15

Since \( a_{l_1} + a_{x+1} = (a_x + a_{l_2}) + a_{x+1} = a_x + (a_{l_2} + a_{x+1}) \in S, a_{l_2} + a_{x+1} \in \{a_{x-1}, a_{j_0}, a_{l_1}\} \).

Since \( (a_x + a_{l_2}) + a_{x-1} = a_x + (a_{l_2} + a_{x-1}) \notin S \), \( a_{l_3} + a_{x-1} \in \{a_x, a_{x+1}\} \cup C \).

Similarly, \( a_{x+1} + a_{l_3} \in \{a_{x-1}\} \cup C; a_{l_1} + a_{x-1} \in \{a_x, a_{x+1}\} \cup C; a_{j_0} + a_{x-1} \in \{a_x, a_{x+1}\} \cup C; a_{l_1} + a_{l_3} \in \{a_{x-1}, a_{x-1}\} \cup C \).

(II.1) If \( a_{x+1} + a_{l_3} = a_{x-1} \) then \( a_{x+1} + a_{l_2} = a_{x+1} + (a_x + a_{l_2}) = (a_x + a_{l_1}) + a_{x-1} \), contradicting \( a_{x+1} + a_{l_2} \in \{a_{x-1}, a_{j_0}, a_{l_1}\} \).

(II.2) If \( a_{x+1} + a_{l_1} = C \) then \( a_{x+1} + a_{l_2} \in C \) (since \( a_{x+1} + a_{l_3} + a_{l_1} = (a_{x+1} + a_{l_1}) + a_{l_3} \notin S \)).

Then \( (a_{x+1} + a_{l_1}) + a_{l_2} = (a_{x+1} + a_{l_2}) + a_{l_1} \notin S \).

Since \( a_{x+1} + a_{l_3} \in C, (a_{x+1} + a_{l_1}) + a_{l_2} = (a_{x+1} + a_{l_2}) + a_{l_3} \notin S \). Uniting \( a_{l_2} + a_{x+1} \in \{a_{x-1}, a_{j_0}, a_{l_1}\} \), we have \( a_{x+1} + a_{l_2} = a_{l_1} \). So \( a_{l_1} + a_{l_2} = (a_{x+1} + a_{l_1} + a_{l_2}) + a_{l_3} = a_{x+1} + (a_{l_1} + a_{l_3}) \notin S \), which implies \( a_{l_1} + a_{l_3} \in \{a_{x+1}\} \cup C \).

So \( a_{l_1} + a_{j_0} = (a_{x+1} + a_{l_2}) + a_{j_0} = a_{x+1} + (a_{l_2} + a_{j_0}) \in S \), which implies \( a_{l_2} + a_{j_0} \in \{a_x\} \cup \{V_x\} \).

Then \( a_{x+1} + (a_{l_1} + a_{j_0}) = (a_{x+1} + a_{l_1}) + a_{j_0} \notin S \), which implies that \( a_{l_1} + a_{j_0} \in \{a_x\} \cup C \).

(II.2.1) If \( a_{l_1} + a_{j_0} \in C \) then \( (a_{l_1} + a_{j_0}) + a_{l_2} = a_{l_1} + (a_{l_2} + a_{j_0}) \notin S \), a contradiction \( a_{l_1} + a_{j_0} \in \{a_x\} \cup \{V_x\} \).

(II.2.2) If \( a_{l_1} + a_{j_0} = a_x \) then \( a_{x+1} + a_{l_2} = (a_{l_1} + a_{j_0}) + a_{l_2} = a_{l_1} + (a_{l_2} + a_{j_0}) \in S \), which implies \( a_{l_2} + a_{j_0} \in \{a_{x-1}, a_{x+1}\} \).

(II.2.2.1) If \( a_{l_2} + a_{j_0} \in C \) then \( (a_{l_2} + a_{j_0}) + a_{l_3} = (a_{l_2} + a_{j_0}) + a_{l_1} = a_{l_3} + a_{j_0} \in \{a_{x-1}, a_{x+1}\} \).

(II.2.2.2) If \( a_{l_2} + a_{j_0} = a_{x+1} \) then \( a_{l_2} + a_{j_0} = a_{x+1} \).

Since \( a_{x-1} + a_{l_1} = (a_{l_2} + a_{j_0}) + a_{l_1} = a_{l_2} + (a_{l_1} + a_{j_0}) = a_{l_2} + a_{x} = a_{l_1}, a_{x-1} + a_{l_1} = a_{l_1}, \) a contradiction.
Thus, Lemma 2.7 holds. □

**Lemma 2.8** If $a_x \in V$ with $|a_x| = \max\{|a| : a \in V\}$, then $a_x + a_p \in C$ for any $a_x a_p \in E$ with $n \geq 8$.

**Proof:** Let $|a_x| = \max\{|a| : a \in V\}$ with $a_x \in V$. Assume that $a_x > 0$ (A similar argument works for $a_x < 0$). By contradiction. Suppose to the contrary that there exist $a_{p_0} \in V$ and $a_{k_0} \in V - \{a_{p_0}, a_x\}$ such that $a_x + a_{p_0} = a_{k_0}$. According to the choice of $a_x$, $a_x + a_{p_0} > 0$ and $a_{p_0} < 0$. Let $V_0 = \{a_{k_0}, a_x\} \cup \overline{V_{k_0}} \cup \overline{a_x}$. Then $a_x a_l \in E$ and $a_{k_0} a_l \in E$ for all $a_l \in V - V_0$. So $a_{k_0} + a_l = (a_x + a_{p_0}) + a_l = (a_x + a_l) + a_{p_0} \in S$. Thus, $a_x + a_l \in V - \{a_x, a_{k_0}, a_{p_0}\}$ with $a_x + a_l > 0$ and $a_l < 0$. Since $n \geq 8$, there exists at least one such vertex $a_l$ above (see Figure 16).

![Figure 16](image.png)

II. Figure 16

On the other hand, by Lemma 2.2, there exists one edge $a_x a_{j_0} \in E$ such that $a_x + a_{j_0} \in C$ for $n \geq 8$. Then $a_x a_j \in E$ for all $a_j \in V - \{a_x\} \cup \overline{a_x}$. So $a_x + a_{j_0} + a_j = (a_x + a_j) + a_{j_0} \not\in S$. Thus, $a_x + a_j \in \{a_{j_0}\} \cup \overline{a_{j_0}} \cup C$ for all $a_j \in V - \{a_x\} \cup \overline{a_x}$.

For all $a_l \in V - V_0$, $a_x + a_l \in \{a_{j_0}\} \cup \overline{a_{j_0}}$. Since $|V_l| \in \{1, 2\}$ for all $a_l \in V$, $n - 6 \leq |V - V_0| \leq |\{a_{j_0}\} \cup \overline{a_{j_0}}| \leq 3$, that is, $n \leq 9$. So we only consider $n = 9$ and $n = 8$. If $|V_l| = 2$ then let $V_l = \{a_i - 1, a_i + 1\}$ for any $a_i \in V$. If $|V_l| = 1$ then let $V_l = \{a_i +iright\}$ for any $a_i \in V$.

**Case 1** $n = 9$

I. If $a_{j_0}$ is an end vertex of $P_n$ then $|\{a_{j_0}\} \cup \overline{a_{j_0}}| = 2$, contradicting $|V - V_0| \geq 3$.

II. If $a_{j_0}$ is not an end vertex of $P_n$ then $|\{a_{j_0}\} \cup \overline{a_{j_0}}| = 3$. So $a_x$ is not an end vertex of $P_n$ (If not, $|\{a_x\} \cup \overline{a_x}| = 2$. So $|V - V_0| \geq 4$, contradicting $|\{a_{j_0}\} \cup \overline{a_{j_0}}| \leq 3$). Thus, only $|\{a_x\} \cup \overline{a_x}| = |\{a_{k_0}\} \cup \overline{a_{k_0}}| = |\{a_{j_0}\} \cup \overline{a_{j_0}}| = 3$.

Note: $n = 9$ and none of the vertices in $\{a_x, a_{j_0}, a_{k_0}\}$ is an end vertex of $P_n$.

If $a_{k_0} a_{j_0} \in E$ then $a_{k_0} + a_{j_0} = (a_x + a_{p_0}) + a_{j_0} = (a_x + a_{j_0}) + a_{p_0} \in S$, contradicting $a_x + a_{j_0} \in C$.  

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If \( a_{k_0}a_{j_0} \notin E \) then there exists one vertex \( a_y \in V - V_0 \) such that \( a_x + a_y = a_{j_0+1} \) with \( a_{j_0}a_{j_0+1} \notin E \) and \( a_{j_0+1} \in V \). So \( a_{k_0} + a_{j_0+1} = a_{k_0} + (a_x + a_y) = (a_x + a_{k_0}) + a_y \in S \), contradicting \( a_x + a_{k_0} \in C \).

**Case 2**: \( n = 8 \)

(1) If \( a_{j_0} \) is an end vertex of \( P_n \) then \( \{|a_{j_0}\} \cup \{a_{j_0+1}\} \). Let \( \overline{V_{j_0}} = \{a_{j_0+1}\} \).

(1.1) If \( a_x \) is the other end vertex of \( P_n \) then we consider the below.

If \( a_{j_0} = a_{k_0} = a_x + a_{p_0} \) then \( |V_0| = 4 \), contradicting \( \{|a_{j_0}\} \cup \{a_{j_0+1}\} \).

If \( a_x + a_{p_0} = a_{k_0} \notin \{|a_{j_0}, a_{j_0+1}\} \) then \( a_{j_0}a_{k_0} \in E \). So \( a_{j_0} + a_{k_0} = a_x + a_{p_0} = (a_x + a_{j_0}) + a_{p_0} \in S \), contradicting \( a_x + a_{j_0} \in C \) (See Figure 17).

![Figure 17](image)

If \( a_{j_0} \neq a_{k_0} \) and \( a_{j_0+1} = a_{k_0} = a_x + a_{p_0} \) then \( |V - V_0| = 3 \), contradicting \( \{|a_{j_0}\} \cup \{a_{j_0+1}\} \). (See Figure 18).

![Figure 18](image)

(1.2) If \( a_x \) is not the other end vertex of \( P_n \) then \( \{|a_x\} \cup \{a_{x+1}\} \). So \( \overline{V_{x+1}} = \{a_{x+1}\} \).

(1.2.1) If \( a_{j_0} = a_{k_1} \) with \( a_{j_0}a_{j_0+1} \notin E \), then there exist two distinct vertices \( a_y, a_{j_0-1} \in V - V_0 \), then \( \{|a_{j_0}, a_y, a_x + a_{j_0-1}\} \notin \{|a_{j_0}, a_{j_0+1}\} \). Since \( a_x + a_{j_0} \in C \), \( \{|a_x, a_{k_0+1}\} \cup \{|a_{x}, a_{j_0+1}\} \notin S \), then \( a_x + a_{k_0+1} \in C \).

Select any vertex \( a_z \in \{|a_y, a_{j_0-1}\} \) and then \( a_{j_0} + a_{k_0+1} = (a_x + a_z) + a_{k_0+1} = (a_x + a_{k_0+1}) + a_z \in S \), contradicting \( a_x + a_{k_0+1} \in C \) (See Figure 19).
(I.2.2) If $V = \{a_{x-1}, a_x, a_x+1, a_{k_0}, a_{k_0+1}, a_{k_0-1}, a_{j_0}, a_{j_0+1}\}$, then $a_{x+1}$ is the other end vertex of $P_n$. So $a_x a_{x+1} \not\in E$. Since $a_x + a_{j_0} = a_{k_0} > 0$, we have $a_x + a_{k_0} \in C$. Since $\{a_x + a_{k_0}, a_x + a_{j_0}\} \subseteq C$, we have $\{a_x + a_{j_0+1}, a_x + a_{k_0-1}\} \subseteq C$. So only $a_x + a_{k_0+1} = a_{k_0}$. Thus, $a_{k_0} + a_{j_0} = (a_x + a_{k_0+1}) + a_{j_0} = (a_x + a_{j_0}) + a_{k_0+1} \in S$, contradicting $a_x + a_{j_0} \in C$ (See Figure 20).

(II) If $a_{j_0}$ is not an end vertex of $P_n$ then $|\{a_{j_0}\} \cup \overline{V_{j_0}}| = 3$.

(II.1) If there exist two distinct vertices $a_{l_1}, a_{l_2} \in V - V_0$ such that $a_x + a_{l_1} = a_{j_0-1} > 0$ and $a_x + a_{l_2} = a_{j_0+1} > 0$ then $a_x + a_{j_0-1} \in C$ and $a_x + a_{j_0+1} \in C$. So $a_{j_0-1} + a_{j_0+1} = (a_x + a_{l_1}) + a_{j_0+1} = (a_x + a_{j_0+1}) + a_{l_1} \in S$, contradicting $a_x + a_{j_0+1} \in C$ (See Figure 21).
(II.2) Let \( \{a_{y_1}, a_{y_2}\} = \{a_{j_0-1}, a_{j_0+1}\} \). If there exists at most one vertex \( a_{y_1} \in \{a_{j_0-1}, a_{j_0+1}\} \) such that \( a_x + a_{l_1} = a_{y_1} > 0 \), then we can consider \( a_{y_1} \) as \( a_{k_0} \) in the following.

(II.2.1) If \( a_x a_{y_2} \in E \) then \( a_x + a_{y_2} \in V \cup C \).

If \( a_x + a_{y_2} \in C \) then \( a_{y_1} + a_{y_2} = (a_x + a_{l_1}) + a_{y_2} = (a_x + a_{y_2}) + a_{l_1} \in S \), contradicting \( a_x + a_{y_2} \in C \).

If \( a_x + a_{y_2} \in V \) and \( a_{l_1}, a_{y_1} \notin E \) then \( a_x + a_{y_2} \in \{a_{j_0}\} \cup V_{j_0} \), a contradiction (See Figure 22).

![Figure 22](image)

If \( a_x + a_{y_2} \in V \) and \( a_{l_1}, a_{y_1} \in E \) then exists one vertex \( a_z \in V - V_0 \) such that \( V - V_0 \) such that \( a_x + a_z \in \{a_{y_1}, a_{y_2}, a_{j_0}\} \). But it is impossible (See Figure 23).

![Figure 23](image)

(II.2.2) If \( a_x a_{y_2} \notin E \) then \( a_x a_{y_1} \in E \) and there exist two distinct vertices \( a_{x_1}, a_{x_2} \in V - V_0 \) such that \( a_x a_{x_1} \in E \) and \( a_x a_{x_2} \in E \). Since \( a_x + a_{l_1} = a_{y_1} \), we have \( \{a_x + a_{x_1}, a_x + a_{x_2}\} = \{a_{y_2}, a_{j_0}\} \). Assume that \( a_x + a_{x_1} = a_{y_2} \). Then \( a_{y_1} + a_{y_2} = a_{y_1} + (a_x + a_{x_1}) = a_{x_1} + (a_x + a_{y_1}) \in S \), contradicting \( a_x + a_{y_1} \in C \) (see Figure 24).

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Thus, Lemma 2.8 holds. □

Lemma 2.9 Let \(|a_x| = \max\{|a| : a \in V\}\) and \(E_x = \{a_xa_i | a_xa_i \in E\}\) with \(a_x \in V\). Then \(a_k + a_l \in C\) for any \(a_k a_l \in E - E_x\) for \(n \geq 7\).

Proof: Let \(|a_x| = \max\{|a| : a \in V\}\) and \(E_x = \{a_xa_i | a_xa_i \in E\}\) with \(a_x \in V\). If \(|V| = 2\) then we may assume that \(V = \{i-1, i+1\}\) for \(i \in V\). Assume \(a_x > 0\) (A similar argument works for \(a_x < 0\)). By lemma 2.3, \(a_x + a_l \in C\) for any \(a_x a_l \in E_x\). For all \(a_k a_l \in E - E_x\), either there exists one vertex in \(\{a_k, a_l\}\) (we may assume \(a_k\)) such that \(a_k a_x \in E\), or \(a_k a_x \notin E\) and \(a_l a_x \notin E\).

Claim 1 If \(a_k\) and \(a_l\) are the end vertices of \(P_n\) then all the sums of the edges adjacent to \(a_k\) or \(a_l\) belong to \(C\).

In fact, if \(a_k\) and \(a_l\) are the end vertices of \(P_n\) then \(a_k + a_l \in E\) and \(d_G(a_k) = d_G(a_l) = n - 2\). By lemma 2.3, \(a_k + a_l \in C\). For all \(a_k a_l \in E - E_x\), \((a_k + a_l) + a_l = a_k + a_l + a_l \notin S\). So \(a_k + a_l \in \{a_k\} \cup V_x\) or \(a_k + a_l \in C\). Then there are at most three edges \(a_k a_{l_1} \in E - E_x\) such that \(a_k + a_{l_1} \in \{a_k\} \cup V_x\) for \(i = 1, 2, 3\) (Since \(|\{a_k\} \cup V_x| \leq 3\). And others belong to \(C\).

If there exist three edges \(a_k a_{l_1} \in E - E_x\) such that \(a_k + a_{l_1} \in V\) then we may assume \(a_k + a_{l_1} = a_x\), with \(a_x \in \{a_k\} \cup V_x\) and \(i \in \{1, 2, 3\}\). Since \(n \geq 7\) and \(d_G(a_k) = n - 2 \geq 5\), there exists one edge \(a_k a_{l_1} \in E - E_x - \{a_k a_{l_1}, a_k a_{l_2}, a_k a_{l_3}\}\) such that \(a_x a_{l_1} \in E\) with \(i_0 \in \{1, 2, 3\}\). Then \(a_k + a_{l_1} \in C\) and \(a_x a_{l_1} \in S\) (see Figure 25).

Figure 25
So $a_{z_0} + a_{t_4} = (a_k + a_{t_0}) + a_{t_4} = (a_k + a_{t_4}) + a_{t_0} \in S$, contradicting the fact $a_k + a_{t_4} \in C$.

It is more easy to get contradictions when there exist two or one edge $a_ka_{t_i} \in E - E_x$ such that $a_k + a_{t_i} \in V$ for $i \in \{1, 2\}$. Thus, all the sums of the edges adjacent to $a_k$ belong to $C$.

A similar argument works for $a_y$.

Thus, Claim 1 holds.

**Claim 2** If $a_h + a_{l'} \in C$ and $a_h + a_{l''} \in C$ with $a_{l'}a_{l''} \in E$ then $a_h + a_l \in C$ for any $a_ha_l \in E$.

In fact, since $a_h + a_{l'} \in C$, $(a_h + a_{l'}) + a_l = (a_h + a_l) + a_{l'} \not\in S$ for all $a_ha_l \in E - \{a_ha_{l'}, a_ha_{l''}\}$. Then $a_h + a_l \in \{a_{l'}\} \cup \{a_{l''}\} \cup C$. Similarly, $a_h + a_l \in \{a_{l'}\} \cup \{a_{l''}\} \cup C$ (see Figure 26).

![Figure 26](image)

Since $a_ha_l \in E$, $\{a_{l'}\} \cup \{a_{l''}\} \cap \{a_{l'}\} \cup \{a_{l''}\} = \emptyset$. Thus, $a_h + a_l \in C$.

Thus, Claim 2 holds.

By Claim 1, if $a_x$ is not an end vertex for $n \geq 7$ then Claim 2 works for any vertex in $V - \{a_x, a_k, a_y\}$ (see Figure 27,28,29).

![Figure 27](image)
If $a_x$ is an end vertex then assume that $a_xa_{x+1} \notin E$ and $a_ka_{k-1} \notin E$ (see Figure 30).

Firstly, Claim 2 works for every vertex in $V - \{a_x, a_{x+1}, a_k, a_{k-1}, a_y\}$. Secondly, Claim 2 works for $a_{x+1}$ and $a_{k-1}$. Thus, $a_k + a_l \in C$ for any $a_ka_l \in E - E_x$ for $n \geq 7$.

Thus, Lemma 2.9 holds. $\Box$
Lemma 2.10 $P_n$ is exclusive for $n \geq 7$. \(\square\)

Lemma 2.11 $\zeta(P_n) \geq 2n - 7$ for $n \geq 7$. \(\square\)

Proof: Let $V = \{b_1, b_2, ..., b_n\}$. Without loss of generality, we can assume that $b_1 < b_2 < ... < b_n$. So $b_1 + b_2 < b_1 + b_3 < b_1 + b_4 < ... < b_1 + b_n < b_2 + b_3 < ... < b_{n-1} + b_n$. Let $C_0 = \{b_1 + b_2, b_1 + b_3, ..., b_1 + b_n, b_2 + b_3, ..., b_{n-1} + b_n\}$. Then there are at most four numbers which are not in $S$, but in $C_0$. On the other hand, the others in $C_0$ are the isolated vertices by Lemma 2.10. Thus, $\zeta(P_n) \geq 2n - 7$ for $n \geq 7$. \(\square\)

Lemma 2.12 $\sigma(P_n) \leq 2n - 7$ for $n \geq 7$.

Proof: Let $V = \{a_1, a_2, ..., a_n\}$ and $S = V \cup C$, where $C$ is the isolated set.

**Case 1:** $n = 2k$ ($k \geq 4$).

$a_i = (i - 1) \times 10 + 1, i = 1, 2, 3, ..., n,$

$c_j = (j + 2) \times 10 + 2, j = 1, 2, 3, ..., n - 3, n - 1, n + 1, n + 2, ..., 2n - 5,$

$C = \{c_1, c_2, ..., c_{n-3}, c_{n-1}, c_{n+1}, c_{n+2}, ..., c_{2n-5}\}$.

Let us verify this labelling is a sum labelling in detail.

(1) The vertices in $S$ are distinct.

(2) For all $i, j \in \{1, 2, ..., n\}$ and $i \neq j$, $a_i + a_j = [(i + j - 4) + 2] \times 10 + 2$. Since $1 \leq i, j \leq n$ and $i \neq j$, $-1 \leq i + j - 4 \leq 2n - 5$. So $a_i a_j \not\in E \iff a_i + a_j \not\in C \iff a_i + a_j \in \{12, 22, 10n + 2, (n + 2) \times 10 + 2\} \iff i + j - 4 \in \{-1, 0, n - 2, n\}$. That is, $i + j - 4 = -1 \iff i + j = 3 \iff (i, j) \in \{(1, 2), (2, 1)\} \iff a_1 a_2 \not\in E$.

Hence, for any $a_i a_j \not\in E$, $a_i + a_j \not\in S$; for any $a_i a_j \in E$, $a_i + a_j \in S$. Therefore, the labelling is a sum labelling of $P_n \cup (2n - 7)K_1$ for $n = 2k$ and $k \geq 3$.

**Case 2:** $n = 2k + 1$ ($k \geq 3$).

$a_i = (i - 1) \times 10 + 1, i = 1, 2, 3, ..., n,$

$c_j = (j + 2) \times 10 + 2, j = 1, 2, 3, ..., n - 3, n - 1, n + 1, n + 2, ..., 2n - 5,$

$C = \{c_1, c_2, ..., c_{n-3}, c_{n-1}, c_n, c_{n+2}, ..., c_{2n-5}\}$ (For example Figure 31).

Let us verify this labelling is a sum labelling in detail.

(1) The vertices in $S$ are distinct.

(2) For all $i, j \in \{1, 2, ..., n\}$ and $i \neq j$, $a_i + a_j = [(i + j - 4) + 2] \times 10 + 2$. Since $1 \leq i, j \leq n$ and $i \neq j$, $-1 \leq i + j - 4 \leq 2n - 5$. So $a_i a_j \not\in E \iff a_i + a_j \not\in C \iff a_i + a_j \in \{12, 22, 10n + 2, (n + 3) \times 10 + 2\} \iff i + j - 4 \in \{-1, 0, n - 2, n + 1\}$. That is, $i + j - 4 = -1 \iff i + j = 3 \iff (i, j) \in \{(1, 2), (2, 1)\} \iff a_1 a_2 \not\in E$.

Hence, for any $a_i a_j \not\in E$, $a_i + a_j \not\in S$; for any $a_i a_j \in E$, $a_i + a_j \in S$. Therefore, the labelling is a sum labelling of $P_n \cup (2n - 7)K_1$ for $n = 2k$ and $k \geq 3$. 
2, \frac{n+3}{2} + 7), ..., (5, n), (n, 2), (2, 1), (1, 3), (3, n - 1), (n - 1, 6), (6, n - 4)\}, ..., (n - 2, 4)\}. \text{So } P_n = a_{\frac{n+3}{2}+1}a_{\frac{n+3}{2}+1}a_{\frac{n+3}{2}+4}a_{\frac{n+3}{2}+7} \ldots a_8a_{n-3}a_n\text{.}

Hence, for any \(a_i a_j \not\in E\), \(a_i + a_j \not\in S\); for any \(a_i a_j \in E\), \(a_i + a_j \in S\). Therefore, the labelling is a sum labelling of \(P_n \cup (2n - 7)K_1\) for \(n = 2k + 1\) and \(k \geq 3\).

\[\begin{align*}
0 &= \zeta(P_n) < \sigma(P_n) = 1; \\
1 &= \zeta(P_5) < \sigma(P_5) = 2; \\
3 &= \zeta(P_6) < \sigma(P_6) = 4; \\
\zeta(P_n) &= \sigma(P_n) = 0, \quad n = 1, 2, 3; \\
\zeta(P_n) &= \sigma(P_n) = 2n - 7, \quad n \geq 7.
\end{align*}\]

Corollary 2.1

\[\begin{align*}
0 &= \zeta(F_n) < \sigma(F_n) = 1; \\
2 &= \zeta(F_5) < \sigma(F_5) = 3; \\
\zeta(F_n) &= \sigma(F_n) = 0, \quad n = 3, 4; \\
\zeta(F_n) &= \sigma(F_n) = 2n - 8, \quad n \geq 7.
\end{align*}\]

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References


