Positive solutions for a four-point boundary value problem with the \( p \)-Laplacian

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Abstract

In this paper, we study the nonlinear four-point boundary value problem with the \( p \)-Laplacian

\[
\begin{cases}
(\phi_p(u'))' + f(t, u(t)) = 0, & 0 < t < 1, \\
u(0) - \alpha u'(\xi) = 0, & u(1) + \beta u'(\eta) = 0,
\end{cases}
\]

where \( \phi_p(s) = |s|^{p-2}s, p > 1, \alpha, \beta > 0, 0 < \xi < \eta < 1 \). By applying a fixed point theorem in cones, sufficient conditions are given for the existence of a positive solution and multiple positive solutions. The interesting point is the Sturm–Liouville-like boundary condition, which was rarely treated until now.

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1. Introduction

Multi-point boundary value problems (BVPs) have been studied extensively in recent years \([2,3,6–16]\). In particular, the existence of positive solutions is of great interest, see \([2,3,10,12,13,15,16]\) and the references therein. In the problems studied, the following boundary conditions

\begin{align*}
  u(0) &= \alpha u(\xi), & u(1) &= \beta u(\eta); \\
  u'(0) &= \alpha u'(\xi), & u'(1) &= \beta u'(\eta); \\
  u(0) &= \alpha u(\xi), & u'(1) &= \beta u'(\eta); \\
  u'(0) &= \alpha u'(\xi), & u'(1) &= \beta u'(\eta),
\end{align*}

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where \( \alpha, \beta > 0, 0 < \xi < \eta < 1 \), have been considered. Let \( \alpha, \beta \neq 1 \) and \( \xi = 0 \) and \( \eta = 1 \), then (1.1) becomes a Dirichlet boundary condition, (1.2) and (1.3) mixed ones, and (1.4) a Neumann one, respectively.

For the second order nonlinear differential equation with a Sturm–Liouville boundary condition

\[
\begin{cases}
\frac{d^2 u}{dt^2} + f(t, u(t)) = 0, & 0 < t < 1, \\
\alpha u(0) - \beta u'(0) = 0, & \gamma u(1) + \delta u'(0) = 0,
\end{cases}
\]

where \( \alpha, \gamma, \beta, \delta \geq 0 \) and \( \alpha \gamma + \alpha \delta + \beta \gamma > 0 \), many results for the existence of positive solutions have been obtained, see [1,4,5,13] and the references therein. However, there are few works devoted to the four-point boundary condition

\[
\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0,
\]

where \( 0 < \xi, \eta < 1 \), see [2,3,6–14,16]. We call (1.6) a Sturm–Liouville-like four-point boundary condition, since (1.6) becomes a Sturm–Liouville boundary condition when \( \xi = 0 \) and \( \eta = 1 \).

In [12], Su, Wei and Wang researched the four point boundary value problem of the singular \( p \)-Laplacian differential equation

\[
\begin{cases}
\phi_p(u'(t))' + a(t) f(u(t)) = 0, & 0 < t < 1, \\
\alpha \phi_p(u(0)) - \beta \phi_p(u'(0)) = 0, & \gamma \phi_p(u(1)) + \beta \phi_p(u'(1)) = 0.
\end{cases}
\]

In this paper, the authors claimed that \( u \in C[0, 1] \cap C^1(0, 1) \) is a solution of (1.7) if and only if \( u \in C[0, 1] \) is a solution of the following integral equation

\[
u(t) = \begin{cases}
\phi_p\left( \frac{\beta}{\alpha} \int_{\xi}^{\sigma} a(r) f(u(r)) \, dr \right) + \int_{0}^{t} \phi_p\left( \int_{s}^{\eta} a(r) f(u(r)) \, dr \right) \, ds, & 0 \leq t \leq \sigma, \\
\phi_p\left( \frac{\delta}{\gamma} \int_{\sigma}^{\eta} a(r) f(u(r)) \, dr \right) + \int_{t}^{1} \phi_p\left( \int_{\sigma}^{s} a(r) f(u(r)) \, dr \right) \, ds, & \sigma \leq t \leq 1,
\end{cases}
\]

where \( \phi > 1 \) satisfies \( \frac{1}{\phi} + \frac{1}{\phi'} = 1 \) and \( \sigma \in [\xi, \eta] \) is such that \( u'(\sigma) = 0 \). Unfortunately, such a claim is incorrect. The results obtained therein should be reconsidered.

Motivated by the works mentioned above, we aim to discuss the four-point boundary value problem

\[
\begin{cases}
\phi_p(u'(t))' + f(t, u(t)) = 0, & 0 < t < 1, \\
u(0) - \alpha u'(0) = 0, & u(1) + \beta u'(1) = 0,
\end{cases}
\]

where \( \phi_p(s) = \int_{0}^{s} p^{-2} \, dx, p > 1, \alpha, \beta > 0, 0 < \xi < \eta < 1 \). In what follows, we always suppose that \( f : [0, 1] \times [0, +\infty) \to [0, +\infty) \) is continuous, and that for each \( x \in R, f(t, x) \neq 0 \) on any subinterval of \( [0, 1] \). The proofs in this paper are based on Krasnosel’skii’s fixed point theorem.

It is well known that the solutions of BVP (1.5) are nonnegative, if \( f \) is nonnegative. But such a conclusion fails to be true for BVP (1.6) even if \( f(t, x) \geq 0 \) holds. Here, we give a simple example for \( p = 2 \). Consider

\[
\begin{cases}
u''(t) + 2 = 0, & 0 < t < 1, \\
u(0) - \nu'(0) = 0, & u(1) + \nu'(\frac{7}{8}) = 0.
\end{cases}
\]

By a direct calculation, we see that \( u(t) = -t^2 + \frac{17}{12} t - \frac{1}{12} \) is the unique solution of (1.9). Unfortunately, \( u(t) \) changes sign on \([0, 1]\).

In order to guarantee the solution’s positivity, it is vital that the maximum of any such solution must lie between \( \xi \) and \( \eta \). To this end, we introduce a special cone in order to apply Krasnosel’skii’s fixed point theorem under suitable conditions imposed on the nonlinear term \( f \). Here, we note that our boundary condition is equivalent to the one of (1.7).

For the convenience of the reader, we state here Krasnosel’skii’s fixed point theorem.

**Theorem 1.1** ([17]). Let \( X \) be a Banach space and \( P \subset X \) a cone. Let \( \Omega_1, \Omega_2 \subset X \) open and bounded, \( 0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \). Assume that \( T : (\overline{\Omega}_2 \setminus \Omega_1) \cap P \to P \) is completely continuous. If one of the following conditions holds

\[
\begin{cases}
u''(t) + f(t, u(t)) = 0, & 0 < t < 1, \\
u(0) - \nu'(0) = 0, & u(1) + \nu'(1) = 0,
\end{cases}
\]
\( \| Tx \| \leq \| x \|, \forall x \in \partial \Omega_2 \cap P, \| Tx \| \geq \| x \|, \forall x \in \partial \Omega_1 \cap P; \)
\( \| Tx \| \leq \| x \|, \forall x \in \partial \Omega_2 \cap P, \| Tx \| \geq \| x \|, \forall x \in \partial \Omega_1 \cap P, \)
then \( T \) has a fixed point theorem in \( (\Omega_2 \setminus \Omega_1) \cap P \).

2. Preliminaries

In this section, we give some lemmas needed for the proof of our main results.

**Lemma 2.1.** Suppose \( e \in L^1[0, 1], e(t) \geq 0, \) and \( e(t) \neq 0 \) on any subinterval of \([0, 1]\). Then BVP
\[
\begin{cases}
(\varphi_p(u'))' + e(t) = 0, & 0 < t < 1, \\
u(0) - au'(\xi) = 0, & u(1) + \beta u'(\eta) = 0
\end{cases}
\]
has a unique solution \( u \in C[0, 1] \cap C^1(0, 1) \). Moreover, this solution can be expressed as
\[
u(t) = \begin{cases}
\alpha \varphi_q \left( \int_0^\sigma e(r)dr \right) + \int_0^t \varphi_q \left( \int_s^\sigma e(r)dr \right) ds, & 0 \leq t \leq \sigma, \\
\beta \varphi_q \left( \int_0^\eta e(r)dr \right) + \int_t^1 \varphi_q \left( \int_s^\sigma e(r)dr \right) ds, & \sigma \leq t \leq 1,
\end{cases}
\]
where \( q > 1 \) is such that \( 1/p + 1/q = 1 \) and \( \sigma \) is the unique solution of the equation
\[
\alpha \varphi_q \left( \int_0^\sigma e(r)dr \right) + \int_0^t \varphi_q \left( \int_s^\sigma e(r)dr \right) ds = \beta \varphi_q \left( \int_0^\eta e(r)dr \right) + \int_t^1 \varphi_q \left( \int_s^\sigma e(r)dr \right) ds.
\]

**Proof.** Firstly, we show that (2.3) has a unique solution. Set
\[
A_1(t) = \alpha \varphi_q \left( \int_0^t e(r)dr \right) + \int_0^t \varphi_q \left( \int_s^t e(r)dr \right) ds,
\]
\[
A_2(t) = \beta \varphi_q \left( \int_t^\eta e(r)dr \right) + \int_0^1 \varphi_q \left( \int_s^\sigma e(r)dr \right) ds.
\]
Obviously, \( A_1 \) is strictly increasing while \( A_2 \) is strictly decreasing. Furthermore, \( A_1(0) < 0 < A_2(0), A_1(1) > 0 > A_2(1) \). So there is a unique \( \sigma \in (0, 1) \) such that \( A_1(\sigma) = A_2(\sigma) \), since both \( A_1(t) \) and \( A_2(t) \) are continuous.

It is easy to verify that \( u \) given by (2.2) is a solution of (2.1). If (2.1) has a solution, denoted by \( u \), then \( (\varphi_p(u'))' = -e(t) \leq 0 \). Since \( u(0) - \alpha u'(\xi) = 0 \) and \( u(1) + \beta u'(\eta) = 0 \), there exists \( \sigma \in (0, 1) \) such that \( u'(\sigma) = 0 \). Integrating the equation of (2.1) on \([t, \sigma]\), we arrive at
\[
\varphi_p(u'(t)) = \int_t^\sigma e(r)dr,
\]
which implies \( u'(t) = \varphi_q(\int_t^\sigma e(r)dr) \). Furthermore,
\[
u(t) = u(0) + \int_0^t u'(s)ds = \alpha u'(\xi) + \int_0^t \varphi_q \left( \int_s^\sigma e(r)dr \right) ds = \alpha \varphi_q \left( \int_\xi^\sigma e(r)dr \right) + \int_0^t \varphi_q \left( \int_s^\sigma e(r)dr \right) ds, \quad 0 \leq t \leq \sigma.
\]
Similarly, we can obtain
\[
u(t) = \beta \varphi_q \left( \int_t^\eta e(r)dr \right) + \int_0^1 \varphi_q \left( \int_s^\sigma e(r)dr \right) ds, \quad \sigma \leq t \leq 1.
\]
Therefore, (2.2) is the unique solution of (2.1). The proof is complete.  \( \square \)
In order to obtain a nonnegative solution, we impose suitable conditions on \( e \). Set
\[
\Gamma = \left( \min \left\{ \frac{\beta q(\eta - \xi)^q - 1}{\xi^q}, \frac{\alpha q(\eta - \xi)^q - 1}{(1 - \eta)^q} \right\} \right)^{\frac{1}{p - 1}},
\]
and
\[
w(t) = \min \left\{ \frac{1}{\eta}, \frac{1}{1 - \xi}(1 - t) \right\}, \quad t \in [0, 1].
\]

**Lemma 2.2.** Let \( e \in C[0, 1] \) satisfy all conditions of Lemma 2.1. Suppose further that
\[
\max_{t \in [0, \xi - \eta, \eta, 1]} e(t) \leq \Gamma \min_{t \in [\xi, 0]} e(t). \quad (2.4)
\]
Then the solution of BVP (2.1) \( u \) is concave, nonnegative, nondecreasing on \([0, \xi]\), and nonincreasing on \([\eta, 1]\). Moreover,
\[
u(t) \geq w(t)\|u\|, \quad t \in [0, 1].
\]

**Proof.** From the proof of Lemma 2.1, we know that there exists \( \sigma \in (0, 1) \) such that \( u'(\sigma) = 0 \). Now we show that \( \sigma \in (\xi, \eta) \). Suppose to the contrary that \( \sigma \in (0, \xi) \). From (2.2) and (2.4), and we have
\[
u(\sigma) = \alpha \varphi_q \left( \int_0^\sigma e(r)dr \right) + \int_0^\sigma \varphi_q \left( \int_r^\sigma e(r)dr \right) ds
\]
\[
< \int_0^\xi \varphi_q \left( \int_r^\xi e(r)dr \right) ds \leq \frac{1}{\xi^q} \varphi_q \left( \max_{t \in [0, \xi]} e(t) \right)
\]
\[
\leq \left( \beta (\eta - \xi)^q - 1 + \frac{1}{\xi^q} (\eta - \xi)^q \right) \varphi_q \left( \min_{t \in [\xi, 0]} e(t) \right)
\]
\[
\leq \beta \varphi_q \left( \int_0^\xi e(r)dr \right) + \int_0^\xi \varphi_q \left( \int_r^\xi e(r)dr \right) ds
\]
\[
< \beta \varphi_q \left( \int_0^\xi e(r)dr \right) + \int_0^\xi \varphi_q \left( \int_r^\xi e(r)dr \right) ds
\]
\[
= \nu(\sigma)
\]
which is a contraction. So \( \sigma \notin (0, \xi) \). Similarly, \( \sigma \notin (\eta, 1) \). The concavity of \( u \) guarantees that \( u \) is nondecreasing on \([0, \xi]\) and nonincreasing on \([\eta, 1]\). Noticing that \( u(0) = \alpha u'(\xi) \geq 0 \), \( u(1) = -\beta u'(\eta) \geq 0 \), we obtain that \( u \) is nonnegative.

Furthermore, we have
\[
\frac{u(t)}{t} \geq \frac{u(\sigma)}{\sigma} \geq \frac{u(\sigma)}{\eta} = \frac{1}{\eta} \|u\|, \quad t \in (0, \sigma],
\]
and
\[
\frac{u(t)}{1 - t} \geq \frac{u(\sigma)}{1 - \sigma} \geq \frac{u(\sigma)}{1 - \xi} = \frac{1}{1 - \xi} \|u\|, \quad t \in [\sigma, 1).
\]
It follows that \( u(t) \geq w(t)\|u\|, \quad t \in [0, 1] \). The proof is complete. \( \square \)

**Remark 2.1.** Condition (2.4) can be weakened. In fact, change (2.4) into
\[
\varphi_q \left( \max_{t \in [0, \xi]} e(t) \right) \leq \frac{1}{\xi^q} \left( \beta (\eta - \xi)^q - 1 \varphi_q \left( \min_{t \in [\xi, 0]} e(t) \right) + \frac{1}{q} (1 - \xi)^q \varphi_q \left( \min_{t \in [\xi, 1]} e(t) \right) \right).
\]
and
\[ \varphi_q \left( \max_{t \in [\xi, \eta]} e(t) \right) \leq \frac{1}{(1 - \eta)^q} \left( \alpha(\eta - \xi)^{q-1} \varphi_q \left( \min_{t \in [\xi, \eta]} e(t) \right) + \frac{1}{q} \eta^q \varphi_q \left( \min_{t \in [0, \eta]} e(t) \right) \right), \]

Lemma 2.2 also holds.

**Remark 2.2.** If condition (2.4) holds, then \( \Gamma \geq 1 \). Obviously, for a given \( \xi, \eta \), such a condition imposes restrictions on \( \alpha, \beta \).

**Lemma 2.3.** Suppose \( e \in C([0, 1], [0, +\infty)) \) satisfies
\[
\min_{t \in [0, \xi]} e(t) > \left( \frac{\beta q(\eta - \xi)^{q-1} + (1 - \xi)^q}{\xi^q} \right)^{p-1} \max_{t \in [\xi, 1]} e(t) \tag{2.5}
\]
or
\[
\min_{t \in [0, \xi]} e(t) > \left( \frac{\alpha q(\eta - \xi)^{q-1} + \eta^q}{(1 - \eta)^q} \right)^{p-1} \max_{t \in [0, \eta]} e(t). \tag{2.6}
\]

Then BVP (2.1) has no positive solutions.

**Proof.** Suppose BVP has a solution \( u \). Then (2.5) implies \( u'(\xi) < 0 \), and (2.6) implies \( u'(\eta) > 0 \). From the boundary condition, we have \( u(0) = \alpha u'(\xi) < 0 \) or \( u(1) = -\beta u'(\eta) < 0 \). So \( u \) is not positive. The proof is complete. \( \square \)

3. Existence of positive solutions

Choose a positive number \( k > 2 \) large enough such that \( \xi, \eta \in [\frac{1}{k}, 1 - \frac{1}{k}] \). First, we introduce some notations. Set
\[
M_R = \max\{f(t, u), t \in [0, \xi] \cup \{\eta, \xi\}, u \in [0, R]\},
\]
\[
N_R = \min\{f(t, u), t \in [\xi, \eta], u \in [w^* R, R]\}, \quad w^* = \min\{w(\xi), w(\eta)\},
\]
\[
f^a = \limsup_{u \to a} \max_{t \in [0, 1]} \frac{f(t, u)}{\varphi_p(u)}, \quad f_a = \liminf_{u \to a} \min_{t \in [0, 1]} \frac{f(t, u)}{\varphi_p(u)}, \quad (a = 0^+ \text{ or } +\infty),
\]
\[
\frac{1}{m} = \max_{\sigma \in [\xi, \eta]} \frac{1}{2} \left( \alpha(\sigma - \xi)^{q-1} + \frac{1}{q} \sigma^q + \beta(\eta - \sigma)^{q-1} + \frac{1}{q} (1 - \sigma)^q \right),
\]
\[
\frac{1}{M} = \min_{\sigma \in [\xi, \eta]} \frac{1}{2} \left( \alpha(\sigma - \xi)^{q-1} + \frac{1}{q} \left( \sigma - \frac{1}{k} \right)^q + \beta(\eta - \sigma)^{q-1} + \frac{1}{q} \left( 1 - \frac{1}{k} - \sigma \right)^q \right).
\]
It is easy to verify that \( m < M \). We note condition (H1) and (H2) here.

(H1) There exists a positive number \( R \) such that \( M_R \leq \Gamma N_{\gamma R} \) for all \( \gamma \in [0, R] \).

(H2) \( M_R \leq \Gamma N_{\gamma R} \) for all \( \gamma > 0 \).

Consider the Banach space \( X = C[0, 1] \) with maximum norm \( ||u|| = \max_{t \in [0, 1]} |u(t)| \). Define a cone \( K \) in \( X \) by
\[
K = \left\{ u \in X : u(t) \geq 0, t \in [0, 1], u \text{ is concave, nondecreasing on } [0, \xi], \text{ and nonincreasing on } [\eta, 1] \right\}.
\]

**Remark 3.1.** If \( u \in K \cap C^1(0, 1) \), then there exists \( \sigma \in [\xi, \eta] \) such that \( u'(\sigma) = 0 \), which is important for the positivity of \( u(t) \). Meanwhile, for all \( u \in K \), \( u(t) \geq w(t)||u||, t \in [0, 1] \).

Define the operator \( T : K \to X \) by
\[
(Tu)(t) = \begin{cases} 
\alpha \varphi_q \left( \int_{\xi}^{\sigma} f(r, u(r))dr \right) + \int_{0}^{t} \varphi_q \left( \int_{s}^{\sigma} f(r, u(r))dr \right) ds, & 0 \leq t \leq \sigma, \\
\beta \varphi_q \left( \int_{\sigma}^{\eta} f(r, u(r))dr \right) + \int_{t}^{1} \varphi_q \left( \int_{\sigma}^{s} f(r, u(r))dr \right) ds, & \sigma \leq t \leq 1
\end{cases} \tag{3.1}
\]
where $\sigma$ is determined by

$$\alpha \varphi_q \left( \int_\xi^\sigma f(r, u(r)) dr \right) + \int_0^\sigma \varphi_q \left( \int_s^\sigma f(r, u(r)) dr \right) ds = \beta \varphi_q \left( \int_\eta^\sigma f(r, u(r)) dr \right) + \int_\sigma^1 \varphi_q \left( \int_s^\sigma f(r, u(r)) dr \right) ds.$$ 

**Lemma 3.1.** Let $(H_1)$ or $(H_2)$ hold. Then $T : \overline{K}_R \to K$ (resp. $T : K \to K$) is completely continuous, where $K_R = \{ x \in K : \|x\| < R \}$.

**Proof.** Suppose $(H_1)$ holds. From Lemma 2.2, $T : \overline{K}_R \to K$ is well defined. $T$ is called completely continuous if it is continuous and maps bounded subsets of $\overline{K}_R$ into relatively compact ones.

Now we show that $T$ is continuous. Let $u_n \to u_0 (n \to +\infty)$ in $\overline{K}_R$. Similarly to Lemma 2.1, for any $u_n$, there exists a unique $\sigma_n \in [\xi, \eta]$ such that $B_{1,n}(\sigma_n) = B_{2,n}(\sigma_n)$, $n = 0, 1, 2, \ldots$, where

$$B_{1,n}(t) = \alpha \varphi_q \left( \int_\xi^{\sigma_n} f(r, u_n(r)) dr \right) + \int_0^{\sigma_n} \varphi_q \left( \int_s^{\sigma_n} f(r, u_n(r)) dr \right) ds,$$

$$B_{2,n}(t) = \beta \varphi_q \left( \int_\eta^{\sigma_n} f(r, u_n(r)) dr \right) + \int_\sigma^{1} \varphi_q \left( \int_s^{\sigma_n} f(r, u_n(r)) dr \right) ds.$$ 

Meanwhile, we can obtain $\sigma_n \to \sigma_0(n \to +\infty)$, $B_{1,n} \to B_{1,0}(n \to +\infty)$, $i = 1, 2$. Let $\sigma_n = \min\{\sigma_n, \sigma_0\}$ and $\bar{\sigma}_n = \max\{\sigma_n, \sigma_0\}$, $n = 1, 2, \ldots$. Obviously, when $t \in \Delta_n = [\sigma_n, \bar{\sigma}_n]$, $t - \sigma_n \to 0$ as $n \to +\infty$. Noticing that

$$\max_{t \in \Delta_n} |B_{i,n}(t) - B_{j,0}(t)| \leq \max_{t \in \Delta_n} |B_{i,n}(t) - B_{i,0}(\sigma_n)| + |B_{j,n}(\sigma_n) - B_{j,0}(\sigma_n)|$$

$$+ \max_{t \in \Delta_n} |B_{j,0}(\sigma_n) - B_{j,0}(t)| \to 0, \quad \text{as } n \to +\infty, i, j = 1, 2, i \neq j,$$

we have

$$\|Tu_n - Tu_0\| = \max \left\{ \|B_{1,n} - B_{1,0}\|_{[0, \bar{\sigma}_n]}, \|B_{2,n} - B_{1,0}\|_{\Delta_n}, \|B_{1,n} - B_{2,0}\|_{\Delta_n}, \|B_{2,n} - B_{2,0}\|_{[\bar{\sigma}_n, 1]} \right\} \to 0, \quad \text{as } n \to +\infty.$$ 

So $T$ is continuous.

It is easy to prove that $T(D)$ is bounded and equi-continuous, where $D \subset \overline{K}_R$ is a bounded set. By the Arzelà–Ascoli theorem, $T(D)$ is relatively compact. So $T : \overline{K}_R \to K$ is completely continuous.

Similarly, when condition $(H_2)$ holds, we can also prove that $T : K \to K$ is completely continuous. The proof is complete. $\Box$

**Theorem 3.2.** Let $(H_1)$ hold. Suppose there exists $0 < r, \rho < R$ such that either $0 < r < \frac{m}{M}\rho < \rho < R$ or $0 < \rho < \bar{w}r < r < R$ and that the following conditions hold.

$(H_3)$ $f(t, u) \leq (\bar{w}r)^p - 1$, for all $t \in [0, 1], u \in [0, \rho]$.

$(H_4)$ $f(t, u) \geq (Mr)^p - 1$, for all $t \in [\frac{1}{r}, 1 - \frac{1}{r}], u \in [\bar{w}r, r]$.

Here $\bar{w} = \min\{w(\frac{1}{r}), w(1 - \frac{1}{r})\}$. Then boundary value problem (1.8) has a positive solution $u \in \overline{K}_R$ such that $\|u\|$ lies between $r$ and $\rho$.

**Proof.** Let $X, K$ be defined as above. Define $T$ on $K_R$ as (3.1). From Lemma 3.1, $T : K_R \to K$ is completely continuous. It is easy to prove that each fixed point of $T$ coincides with a positive solution of BVP (1.8). So, it is enough to show that $T$ has at least one positive fixed point. Without loss of generality, we suppose that $r < \frac{m}{M}\rho < \rho$.

Set

$$K_r = \{ u \in K : \|u\| < r \}, \quad K_\rho = \{ u \in K : \|u\| < \rho \}.$$ 

For $u \in K_\rho$, $0 \leq u(t) \leq \|u\| = \rho$, $t \in [0, 1]$. By $(H_3)$, we have

$$\|Tu\| = (Tu)(\sigma) = \frac{1}{2} \left( \alpha \varphi_q \left( \int_\xi^\sigma f(r, u(r)) dr \right) + \int_0^\sigma \varphi_q \left( \int_s^\sigma f(r, u(r)) dr \right) ds \right.$$
has at least one positive solution $u$. For any $\alpha > 0$,
Lemma 3.1 leads to the conclusion that BVP (1.8) has a positive solution $u$.

Then boundary value problem (1.8) holds. Suppose that the following conditions hold.

(H5) $f^0 < \varphi_p(m)$.

(H6) $f_\infty > \varphi_p(M/\tilde{w})$.

Then boundary value problem (1.8) has at least one positive solution $u \in K$.

Proof. For any $\varepsilon > 0$ small enough, because $f^0 < \varphi_p(m)$, there exists $\rho > 0$ such that

\[
f(t, u) \leq (\varphi_p(m) + \varepsilon) \varphi_p(u) \leq (\varphi_p(m) + \varepsilon) \varphi_p(\rho), \quad t \in [0, 1], u \in [0, \rho].
\]

which implies that condition (H5) holds.

Similarly, from $f_\infty > \varphi_p(M/\tilde{w})$, there exists $r > 0$ such that \(\tilde{w}r > \rho\) and

\[
f(t, u) \geq \left( \varphi_p \left( \frac{M}{\tilde{w}} \right) - \varepsilon \right) \varphi_p(u), \quad t \in [0, 1], u \in [\tilde{w}r, +\infty).
\]

So,

\[
f(t, u) \geq \left( \varphi_p \left( \frac{M}{\tilde{w}} \right) - \varepsilon \right) \varphi_p(\tilde{w}r), \quad t \in \left[ \frac{1}{k}, 1 - \frac{1}{k} \right], u \in [\tilde{w}r, r]
\]

implies that condition (H6) is satisfied. A similar argument to those of Lemma 3.1 leads to the conclusion that BVP (1.8) has a positive solution $u \in K$. □
**Theorem 3.4.** Let \( (H_2) \) hold. Suppose the following conditions hold.

\[(H_7)\] \( f^\infty < \varphi_p(m) \),
\[(H_8)\] \( f_0 > \varphi_p\left(\frac{M}{w}\right) \).

Then boundary value problem \((1.8)\) has at least one positive solution \( u \in K \).

**Proof.** Let \( \varepsilon > 0 \) small enough, then because \( f_0 > \varphi_p\left(\frac{M}{w}\right) \), there exists \( r > 0 \) such that

\[ f(t, u) \geq \left( \varphi_p\left(\frac{M}{w}\right) - \varepsilon \right) \varphi_p(u), \quad t \in [0, 1], u \in [0, r]. \]

Thus,

\[ f(t, u) \geq \left( \varphi_p\left(\frac{M}{w}\right) - \varepsilon \right) \varphi_p(\tilde{w}r), \quad t \in \left[\frac{1}{k}, 1 - \frac{1}{k}\right], u \in [\tilde{w}r, r], \]

which implies that condition \((H_4)\) holds.

By condition \((H_7)\), there exists \( \rho^* > 0 \) such that

\[ f(t, u) \leq (\varphi_p(m) + \varepsilon)\varphi_p(u), \quad t \in [0, 1], u \in [\rho^*, +\infty). \]

(1) If \( f \) is bounded, then there exists \( L \) such that \( f(t, u) \leq L, t \in [0, 1], u \in [0, +\infty) \). Choose \( \rho > \rho^* \) such that \( \rho > \frac{M}{m}r \) and \( \varphi_p(m\rho) \geq L \). Then

\[ f(t, u) \leq L \leq (\varphi_p(m) + \varepsilon)\varphi_p(\rho), \quad t \in [0, 1], u \in [0, \rho]. \]

(2) If \( f \) is unbounded, then there exists \( \rho, u_0 \) such that \( \rho > u_0 > \rho^*, \rho > \frac{M}{m}r \) and \( f(t, u) \leq f(t, u_0), t \in [0, 1], u \in [0, \rho^*] \). Therefore,

\[ f(t, u) \leq f(t, u_0) \leq (\varphi_p(m) + \varepsilon)\varphi_p(u_0) \]

\[ \leq (\varphi_p(m) + \varepsilon)\varphi_p(\rho), \quad t \in [0, 1], u \in [0, \rho^*], \]

and

\[ f(t, u) \leq (\varphi_p(m) + \varepsilon)\varphi_p(u) \]

\[ \leq (\varphi_p(m) + \varepsilon)\varphi_p(\rho), \quad t \in [0, 1], u \in [\rho^*, \rho]. \]

In any case, \( f(t, u) \leq (\varphi_p(m) + \varepsilon)\varphi_p(\rho), t \in [0, 1], u \in [0, \rho] \). So by **Theorem 3.2**, we complete the proof. \( \square \)

**Corollary 3.5.** Let \( (H_2), (H_3) \) hold. Suppose \((H_6)\) or \((H_8)\) holds. Then boundary value problem \((1.8)\) has a positive solution \( u \in K \).

**Corollary 3.6.** Let \( (H_2), (H_4) \) hold. Suppose \((H_5)\) or \((H_7)\) holds. Then boundary value problem \((1.8)\) has a positive solution \( u \in K \).

Next, we present some results for the existence of at least two positive solutions of BVP \((1.8)\). The proofs are standard, and are hence omitted here.

**Theorem 3.7.** Let \( (H_2) \) hold. Suppose there exists \( \rho > 0 \) such that conditions \((H_3), (H_6)\) and \((H_8)\) hold. Suppose further that \( x \neq T x, x \in \partial \Omega_{\rho} \cap K \). Then, BVP \((1.8)\) has at least two positive solutions \( u_1, u_2 \) in \( K \) such that \( u_1 \neq u_2, \|u_1\| \leq \rho \leq \|u_2\| \).

**Theorem 3.8.** Let \( (H_2) \) hold. Suppose there exists \( r > 0 \) such that conditions \((H_4), (H_5)\) and \((H_7)\) hold. Suppose further that \( x \neq T x, x \in \partial \Omega_{r} \cap K \). Then BVP \((1.8)\) has at least two positive solutions \( u_1, u_2 \) in \( K \) such that \( u_1 \neq u_2, \|u_1\| \leq r \leq \|u_2\| \).

**Theorem 3.9.** Let \( (H_2) \) hold. Suppose there exists \( \rho \) such that one of the following conditions holds.

\[(H_9)\] \( f(t, u) > (Mr)^{p-1}, \) for all \( t \in [\frac{1}{k}, 1 - \frac{1}{k}], u \in [\tilde{w}r, \rho] \) and \( f^0 = 0, f^\infty = 0. \)
\[(H_{10})\] \( f(t, u) < (mp)^{p-1}, \) for all \( t \in [0, 1], u \in [0, \rho] \) and \( f_0 = +\infty, f_\infty = +\infty. \)

Then BVP \((1.8)\) has at least two positive solutions \( u_1, u_2 \) in \( K \) such that \( \|u_1\| < \rho < \|u_2\| \).
4. Examples

In this section, we give some explicit examples to illustrate our main results.

Example 1. Consider the following four-point boundary value problem with a $p$-Laplacian

$$
\begin{cases}
\left( \frac{u'}{\sqrt{|u|}} \right)' + h(t) \left( \frac{8}{13} + \frac{u^2}{7 + 7u^2} \right) = 0, & 0 < t < 1, \\
u(0) - u' \left( \frac{1}{4} \right) = 0, & u(1) + 2u' \left( \frac{1}{2} \right) = 0,
\end{cases}
$$

(4.1)

where $0 < l \leq h(t) \leq L, \ t \in [0, 1]$ and

$$h^* = \max \left\{ h(t), \ t \in \left[ 0, \frac{1}{4} \right] \cup \left[ \frac{1}{2}, 1 \right] \right\} \leq h_* = \min \left\{ h(t), \ t \in \left[ \frac{1}{4}, \frac{1}{2} \right] \right\}.$$

Conclusion. BVP (4.1) has at least one positive solution.

Proof. Obviously,

$$p = \frac{3}{2}, \ \alpha = 1, \ \xi = \frac{1}{4}, \ \beta = 2, \ \eta = \frac{1}{2}, \ w(t) = \min \left\{ 2t, \frac{4}{3}(1 - t) \right\}, \ t \in [0, 1].$$

Choose $k = 4$, then $[\frac{1}{4}, \frac{1}{2}] \subset \left[ \frac{1}{4}, \frac{3}{4} \right]$. By a direct calculation, we have

$$q = 3, \ \Gamma = \sqrt{\frac{13}{8}}, \ m = \frac{96}{13}, \ M = \frac{448}{3}, \ w^* = \frac{1}{2}, \ \tilde{w} = \frac{1}{3}.$$

Furthermore, for any $R > 0$,

$$M_R = \max \left\{ h(t) \left( \frac{8}{13} + \frac{u^2}{7 + 7u^2} \right), \ t \in \left[ 0, \frac{1}{4} \right] \cup \left[ \frac{1}{2}, 1 \right], \ u \in [0, R] \right\}$$

$$\leq h^* \left( \frac{8}{13} + \frac{R^2}{7 + 7R^2} \right) \leq \frac{69}{91} < h_* \sqrt{\frac{8}{13}} = \frac{8}{13} \Gamma h_*$$

$$\leq \frac{8}{13} \sqrt{\frac{R^2}{7 + 7R^2}} \leq \Gamma \min \left\{ h(t) \left( \frac{8}{13} + \frac{u^2}{7 + 7u^2} \right), \ t \in \left[ \frac{1}{4}, \frac{1}{2} \right], \ u \in \left[ \frac{1}{2} R, R \right] \right\}$$

$$\leq \Gamma N_R.$$

Next, we show that the conditions (H7) and (H8) hold. In fact, because

$$f(t, u) = h(t) \left( \frac{8}{13} + \frac{u^2}{7 + 7u^2} \right) \frac{1}{\sqrt{u}}$$

and

$$\left( \frac{8}{13} + \frac{u^2}{7 + 7u^2} \right) \frac{1}{\sqrt{u}} \to 0(+\infty), \ \text{as} \ u \to +\infty (u \to 0),$$

we obtain that $f^\infty = 0 < \varphi_p(m)$ and $f_0 = +\infty > \varphi_p(3M)$ hold.

By Theorem 3.4, (4.1) has at least one positive solution. □

Example 2. Consider the four-point boundary value problem of second order differential equation

$$
\begin{cases}
u'' + (-t^2 + t + 2) \left( \frac{1}{2} + u^2 \right) = 0, & 0 < t < 1, \\
u(0) - u' \left( \frac{1}{3} \right) = 0, & u(1) + u' \left( \frac{2}{3} \right) = 0.
\end{cases}
$$

(4.2)
Conclusion. BVP (4.2) has at least two positive solutions.

Proof. Obviously, \( p = q = 2, \alpha = \beta = 1, \xi = \frac{1}{3}, \eta = \frac{2}{3} \). Set \( k = 3 \). By a direct calculation, we have

\[
\gamma = 7, \quad m = \frac{36}{11}, \quad w^* = \tilde{w} = \frac{1}{2}.
\]

For any \( R > 0 \), one has

\[
M_R = \max \left\{ (-t^2 + t + 2) \left( \frac{1}{2} + u^2 \right), t \in \left[ 0, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, 1 \right], u \in [0, R] \right\}
\leq \frac{20}{9} \left( \frac{1}{2} + R^2 \right) \leq \frac{35}{9} (2 + R^2)
\leq 7 \min \left\{ (-t^2 + t + 2) \left( \frac{1}{2} + u^2 \right), t \in \left[ \frac{1}{3}, \frac{2}{3} \right], u \in \left[ \frac{1}{2} R, R \right] \right\}
\leq \gamma N_R.
\]

Meanwhile, choose \( \rho = \frac{8}{11} \), then we have

\[
f(t, u) \leq \max \left\{ (-t^2 + t + 2) \left( \frac{1}{2} + u^2 \right), t \in [0, 1], u \in [0, \rho] \right\}
\leq \frac{9 + 18 \rho^2}{8} < \frac{36 \rho}{11} = m \rho.
\]

\[
f_0 = \liminf_{u \to 0} \min_{t \in [0, 1]} \frac{(-t^2 + t + 2) \left( \frac{1}{2} + u^2 \right)}{u} = 2 \liminf_{u \to 0} \frac{\frac{1}{2} + u^2}{u} = +\infty,
\]

\[
f_\infty = \liminf_{u \to +\infty} \min_{t \in [0, 1]} \frac{(-t^2 + t + 2) \left( \frac{1}{2} + u^2 \right)}{u} = 2 \liminf_{u \to +\infty} \frac{\frac{1}{2} + u^2}{u} = +\infty.
\]

By Theorem 3.9, (4.2) has at least two positive solutions. \( \square \)

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